Equivariant K-theory for proper actions of non-compact Lie groups

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November 2, 2010

Abstract

Generalizing a construction of Lück and Oliver [9], we define a good equivariant cohomology theory on the category of proper G-CW complexes when G is an arbitrary Lie group (possibly non-compact). This is done by constructing an appropriate classifying space that arises from a $\Gamma - G$-space. It is proven that this theory effectively generalizes Segal's equivariant K-theory when G is compact.

1 Introduction

1.1 The problem

Topological K-theory is a generalized cohomology theory which was developed by Atiyah and Hirzebruch in the early 1960's: starting from a topological space X, one looks at the monoid of isomorphism classes of (complex) finite-dimensional vector bundles bundles over X, and after using a standard algebraic construction (the Grothendieck group associated to a commutative monoid), one recovers an abelian group $\mathbb{K}(X)$, called the K-theory of X, and extends it to a functor $\mathbb{K}(-)$.

This was generalized to spaces by Segal [13] by replacing vector bundles with G-vector bundles. It is now well known that his construction gives rise to a good equivariant cohomology theory in the case of actions of a *compact* group on compact spaces or on G-CW-complexes, and that Bott periodicity still holds.

However, it is still possible to *define* the equivariant K-theory group $\mathbb{K}_G(X)$ whenever X is a G-space. Some properties [8] of $\mathbb{K}_G(-)$ still hold in the general case of a Lie group G acting properly on G-CW-complexes (e.g. two G-homotopic maps have the same image by the functor $\mathbb{K}_G(-)$). However, even in simple cases, $\mathbb{K}_G(-)$ is not a good cohomology theory (excision may fail, cf. [12] and [8]).

A first positive generalization to non-compact groups was given by Phillips using tools from function analysis [12]. However, it would seem reasonable to

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define equivariant K-theory by means of homotopy theory. In the following paper, we will show that a construction of Lück and Oliver that was featured in [9] for discrete groups can be generalized to an arbitrary Lie group. In a following paper, we will relate our equivariant cohomology theory to Phillips' [15].

1.2 Structure of the paper

Let G be a Lie group and F denote one of the fields $\mathbb{R}$ or $\mathbb{C}$. What we want to construct is a good equivariant cohomology theory, which will be written $KF_G(-)$ for convenience, defined on a subcategory of the one of proper G-spaces which contains at least the category of finite proper G-CW-complexes (see the next paragraph for the definitions). For such a theory to deserve the label "equivariant K-theory", we impose a set of conditions. First of all, we want to have product maps, i.e. natural homomorphisms $KF_G(X) \otimes KF_H(Y) \longrightarrow KF_{G \times H}(X \times Y)$. We also want to have Bott homomorphisms (depending on F) that are linear with respect to products, and we want to have Bott periodicity. We finally want a connection between Segal's "naive" equivariant K-theory and our functors. More precisely, we want to have a natural transformation $\mathbb{K}F_G(-) \longrightarrow KF_G(-)$ that yields isomorphisms for the G-spaces of the type $(G/H) \times Y$, where H is a compact subgroup of G, and Y is a reasonable space on which G acts trivially (say a compact space, a CW-complex, or a finite CW-complex). When Y is a sphere, we recover the equality of the so-called "coefficients" of our equivariant K-theory with those of Segal's. The last condition is that the various natural transformations $\mathbb{K}F_G(-) \longrightarrow KF_G(-)$ should be compatible with product maps, and it will follow that they are compatible with the Bott homomorphisms.

The first step (Section 2) consists in constructing a classifying space for the functor $\mathbb{V}\mathrm{ect}_G^F(-)$ which maps every G-CW-complex X to the monoid of isomorphism classes of finite dimensional G-vector bundles over X. In Section 3, we will construct a $\Gamma - G$-space $\underline{\mathrm{Vec}}_G^{F,\infty}$ such that $\mathrm{Vec}_G^{F,\infty}$ has the equivariant homotopy type of that classifying space. The G-space $KF_G^{[\infty]} := \Omega B \, \mathrm{Vec}_G^{F,\infty}$ will then be used in Section 4 as a classifying space to define K-theory $KF_G^*(-)$ in negative degrees. Following Lück and Oliver, product structures and Bott homomorphisms are constructed and then used to define equivariant K-theory in positive degrees. In Section 5, we will finally show that our construction generalizes both Segal's and Lück-Oliver's.

During the course of the construction, we will also define two other classifying spaces along the way: one that is naturally suited for finite-dimensional G-Hilbert bundles and the other for finite-dimensional G-simi-Hilbert bundles (i.e. vector bundle with an added structure that is related to the group of similarities). They will be needed in [15] to relate our equivariant K-theory with Phillips'.

A final word: this work features two theorems with very technical proofs. So as not to distract the reader, we have relegated those proofs in Sections A and B of the appendix. Section C is meant to set things straight on a common misconception on Γ-spaces.

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1.3 The main framework

1.3.1 G-CW-complexes

A G-space X is called a G-**CW-complex** when it is obtained as the direct limit of a sequence $(X_{(n)})_{n\in\mathbb{N}}$ of subspaces for which there exists, for every $n \in \mathbb{N}$, a set I_n, a family $(H_i)_{i\in I_n}$ of closed subgroups of G and a push-out square

$$
\begin{array}{ccc}
\coprod_{i\in I_n} (G/H_i) \times S^{n-1} & \longrightarrow & \coprod_{i\in I_n} (G/H_i) \times D^n \\
\downarrow & & \downarrow \\
X_{(n-1)} & \longrightarrow & X_{(n)}
\end{array}
$$

in the category of G-spaces (where we have a trivial action of G on both the $(n-1)$-sphere S^{n-1} and the closed n-disk D^n), with the convention that $X_{-1} = \emptyset$.
The spaces $(G/H_i) \times \overset{\circ}{D}{}^n$ are called the equivariant cells (or G-cells) of X. A G-CW-complex is **proper** when all its isotropy subgroups are compact, i.e. all the groups H_i in the preceding description are compact. Relative G-CW-complexes are defined accordingly. However, by a **proper** relative G-CW-complex, we mean a relative G-CW-complex (X, A) such that $X \smallsetminus A$ is a proper G-space (whereas X itself may not be proper).

Given a topological group G, a pair of G-spaces (X, A) is said to be a G-**CW-pair** when A and X are G-CW-complexes and (X, A) is a relative G-CW-complex. A G-CW-pair (X, A) is said to be **proper** when X is a proper G-space (i.e. its isotropy groups are compact subgroups of G).

A **pointed proper** G-**CW-complex** is a relative G-CW-complex $(X, *)$, with $*$ a point, such that the G-space $X \smallsetminus *$ is proper. Notice that whenever (X, A) is a relative G-CW-complex such that $X \smallsetminus A$ is proper, the G-space X/A inherits a natural structure of pointed proper G-CW-complex.

1.3.2 G-fibre bundles

Let G be a topological group. Given a G-space X, we call **pseudo-G-vector bundle** (resp. G-**vector bundle**) over X the data consisting of a pseudo-vector bundle (resp. a vector bundle) $p : E \to X$ over X and of a (left) G-action on E, such that p is a G-map, and, for all $g \in G$ and $x \in X$, the map $E_x \to E_{g.x}$ induced by the G-action on E is a linear isomorphism.

Given an integer $n \in \mathbb{N}$ and a G-space X, $\mathbb{V}\mathrm{ect}_G^{F,n}(X)$ will denote the set of isomorphism classes of n-dimensional G-vector bundles over X. Accordingly, $\mathbb{V}\mathrm{ect}_G^F(X)$ will denote the abelian monoid of isomorphism classes of finite-dimensional G-vector bundles over X.

Given another topological group H, a (G, H)-**principal bundle** is an H-principal bundle $\pi : E \to X$ with structures of G-spaces on E and X for which π is a G-map and $\forall (g, h, x) \in G \times H \times E,\ g.(x.h) = (g.x).h$. Notice then that we recover a structure of $(G \times H^{\mathrm{op}})$-space on E for which the isotropy subgroups are closed subgroup which intersect $\{1\} \times H^{\mathrm{op}}$ trivially.

1.3.3 The category of compactly-generated G-spaces

Let G be a topological group. A G**-pointed k-space** consists of a G-space which is a k-space (i.e. compactly-generated and Hausdorff[1]) together with a point in it which is fixed by the action of G.

The category $CG_G^\bullet$ is the one whose objects are the G-pointed k-spaces and whose morphisms are the pointed G-maps. The category $CG_G^{h\bullet}$ is the category with the same objects as $CG_G^\bullet$, and whose morphisms are the equivariant pointed homotopy classes of G-maps between objects (i.e. $CG_G^{h\bullet}$ it is the homotopy category of $CG_G^\bullet$). Given two G-spaces (resp. two pointed G-spaces) X and Y, we let $[X,Y]_G$ (resp. $[X,Y]_G^\bullet$) denote the set of equivariant homotopy classes of G-maps (resp. pointed G-maps) from X to Y.

Let $f : X \to Y$ be a morphism in CG_G and $\mathcal{F}$ be a set of subgroups of G. We say that f is an $\mathcal{F}$**-weak equivalence** when the restriction $f^H : X^H \to Y^H$ is a weak equivalence for every $H \in \mathcal{F}$. We say that f is a G**-weak equivalence** when f is a $\mathcal{K}$-weak equivalence for the set $\mathcal{K}$ of all *compact* subgroups of G. Given a set $\mathcal{F}$ of subgroups of G, every morphism that is equivariantly homotopic to an $\mathcal{F}$-weak equivalence is itself an $\mathcal{F}$-weak equivalence.

We finally define W_G as the class of morphisms in $CG_G^{h\bullet}$ which have G-weak equivalences as representative maps. We can then consider the category of fractions $CG_G^{h\bullet}[W_G^{-1}]$, with its usual universal property. The following properties are then folklore and will be used throughout the paper (see [9] for proofs):

Proposition 1.1. *Let G be a Lie group, $\mathcal{F}$ a family of subgroups of G stable by conjugation and $Y \xrightarrow{f} Y'$ an $\mathcal{F}$-weak equivalence. Then, for every relative G-CW-complex (X, A) such that all the isotropy subgroups of $X \smallsetminus A$ belong to $\mathcal{F}$, and for every commutative square*

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha_0} & Y \\
i \downarrow & & \downarrow f \\
X & \xrightarrow{\alpha} & Y'
\end{array}
$$

in CG_G, there exists a G-map $\tilde{\alpha} : X \to Y$ such that $\tilde{\alpha} \circ i = \alpha_0$ and $[f \circ \tilde{\alpha}] = [\alpha]$ in CG_G^h. Moreover, this map is unique up to an equivariant homotopy rel. A.

Proposition 1.2. *Let $Y \xrightarrow{f} Y'$ be a G-weak equivalence between pointed G-spaces.*
Then, for every proper pointed G-CW-complex X, the map f induces a bijection

$$
f_* : [X,Y]_G^\bullet \longrightarrow [X,Y']_G^\bullet.
$$

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[1]For Section 4.5, and in general whenever smash products are concerned, one should loosen up this definition and define k-spaces as topological spaces that are compactly-generated and *weak-Hausdorff*. The reader will check this bears no additional complexity.
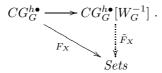
Corollary 1.3. *Let $\mathcal{F}$ be a class of subgroups of G, and $Y \xrightarrow{f} Y'$ an $\mathcal{F}$-weak equivalence between G-spaces. Then, for every G-CW-complex X whose isotropy subgroups all belong to $\mathcal{F}$, the map f induces a bijection:*

$$f_* : [X, Y]_G \longrightarrow [X, Y']_G.$$

Corollary 1.4. *For every proper pointed G-CW-complex X, the functor*

$$F_X : \begin{cases} CG_G^{h\bullet} & \longrightarrow Sets \\ Y & \longmapsto [X, Y]_G^\bullet \end{cases}$$

factorizes through

$$CG_G^{h\bullet} \longrightarrow CG_G^{h\bullet}[W_G^{-1}] \ .$$

$$\underset{F_X}{\searrow} \quad \vdots \, \tilde{F}_X$$

$$Sets$$

Remark 1. We may replace $CG_G^{h\bullet}$ by the category H-$G^{h\bullet}$ whose objects are compactly-generated G-spaces which have an equivariant H-space structure, and the morphisms are equivariant pointed homotopy classes of continuous morphisms of equivariant H-spaces, i.e. $X \xrightarrow{f} Y$ is such a morphism if and only if it is continuous, equivariant, pointed, and the square

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\ f \times f\ } & Y \times Y \\
{\scriptstyle \times_X} \big\downarrow & & {\scriptstyle \times_Y} \big\downarrow \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

is commutative in $CG_G^{h\bullet}$. Let $W_{\text{H-G}}$ denote the class of morphisms in H-$G^{h\bullet}$ which are G-weak equivalences. Then, for every pointed G-CW-complex X, we recover a functor

$$G_X : \begin{cases} \text{H-}G^{h\bullet} & \longrightarrow \mathcal{G}r \\ Y & \longmapsto [X, Y]_G^\bullet. \end{cases}$$

Corollary 1.5. *For every pointed proper G-CW-complex X, the functor G_X factorizes through:*

$$H\text{-}G^{h\bullet} \longrightarrow H\text{-}G^{h\bullet}[W_{H\text{-}G}^{-1}]$$

$$\underset{G_X}{\searrow} \quad \vdots \, \tilde{G_X}$$

$$\mathcal{G}r.$$

1.3.4 Γ-spaces

The simplicial category is denoted by Δ (cf. [2]). Recall that the category Γ (see [14]) has the finite sets as objects, a morphism from S to T being a map

from $\mathcal{P}(S)$ to $\mathcal{P}(T)$ which preserves disjoint unions (with obvious composition of morphisms); this is equivalent to having a map f from S to $\mathcal{P}(T)$ such that $f(s) \cap f(s') = \emptyset$ whenever $s \neq s'$.

For every $n \in \mathbb{N}$, we set $\mathbf{n} := \{1, \ldots, n\}$ and $[n] := \{0, \ldots, n\}$. Recall the canonical functor $\Delta \to \Gamma$ obtained by mapping $[n]$ to $\mathbf{n}$ and the morphism $\delta : [n] \to [m]$ to

$$\begin{cases} \mathbf{n} & \longrightarrow \mathcal{P}(\mathbf{m}) \\ k & \longmapsto \{j \in \mathbb{N} : \{\delta(k-1) < j \leq \delta(k)\}. \end{cases}$$

By a Γ-space, we mean a *contravariant* functor $\underline{A} : \Gamma \to \mathrm{CG}$ such that $\underline{A}(\mathbf{0})$ is a well-pointed contractible space. The space $\underline{A}(\mathbf{1})$ is then simply denoted by A. We say that $\underline{A}$ is a **good** Γ-space when, in addition, for all $n \in \mathbb{N}^*$, the continuous map $\underline{A}(\mathbf{n}) \to \prod_{i=1}^{n} A$, induced by all morphisms $\mathbf{1} \to \mathbf{n}$ which map 1 to $\{i\}$, is a homotopy equivalence. From now on, when we talk of Γ-spaces, we will actually mean good Γ-spaces.

When $\underline{A}$ is a Γ-space, composition with the previously defined functor $\Delta \to \Gamma$ yields a simplicial space, which we still write $\underline{A}$, and we can take its thick geometric realization (as defined in appendix A of [14]), which we write $B A$. Since $\underline{A}(\mathbf{0})$ is well-pointed and contractible, we have a map $A \to \Omega B \underline{A}$ that is "canonical up to homotopy". Recall that we have an H-space structure on A by composing the map $\underline{A}(\mathbf{2}) \to A$ induced by $\begin{cases} \{1\} & \to \mathcal{P}(\mathbf{2}) \\ 1 & \mapsto \{1, 2\} \end{cases}$ and a homotopy inverse of the map $\underline{A}(\mathbf{2}) \to A \times A$ mentioned earlier. Under suitable assumptions on A, one may prove (cf. § 4 of [14]) that the map $A \to \Omega B A$ is in some sense the "group completion" of the H-space A.

Here, we will be dealing with equivariant Γ-spaces, or $\Gamma - G$-**spaces**: given a topological group G, a $\Gamma - G$-space is a contravariant functor $\underline{A} : \Gamma \to \mathrm{CG}_G$ such that:

(i) $\underline{A}(\mathbf{0})$ is equivariantly well-pointed and equivariantly contractible;

(ii) For any $n \in \mathbb{N}^*$, the canonical map $\underline{A}(\mathbf{n}) \to \prod_{i=1}^{n} A$ is an equivariant homotopy equivalence.

Notice that this definition should be more constraining than the one featured in [9]. Moreover, whenever H is a subgroup of G and $\underline{A}$ is a $\Gamma - G$-space, the contravariant functor $\underline{A}^H : \Gamma \to CG$ obtained by restricting $\underline{A}$ to the fixed point sets for H is in fact a Γ-space. When $\underline{A}$ is a $\Gamma - G$-space, we may define as before a G-map $A \to \Omega B A$ which is "canonical up to homotopy" and is a "group completion" of the equivariant H-space A.

1.4 Additional definitions and notation

We set $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{R}_+^* := \{t \in \mathbb{R} : t > 0\}$. The standard segment is denoted by $I := [0, 1]$.

By F, we will always denote one of the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Given two vector spaces E and E' over F, $L(E, E')$ will denote the set of linear maps from E to E'. When $n \in \mathbb{N}$ and $F = \mathbb{R}$ or $\mathbb{C}$, we denote by $U_n(F)$ the unitary subgroup of $\mathrm{GL}_n(F)$, and by $\mathrm{GU}_n(F) = F^* U_n(F)$ the subgroup of similarities of F^n.

Let $k \in \mathbb{N}^*$ and $F = \mathbb{R}$ or $\mathbb{C}$. Then F^k comes with a canonical structure of Hilbert space. We have a canonical sequence of isometries $F^1 \hookrightarrow F^2 \hookrightarrow \ldots \hookrightarrow F^k \hookrightarrow F^{k+1} \hookrightarrow \ldots$, where $F^k \hookrightarrow F^{k+1}$ maps $(x_1, \ldots, x_k)$ to $(x_1, \ldots, x_k, 0)$. We let $F^{(\infty)}$ denote the corresponding direct limit $\varinjlim_{k \in \mathbb{N}^*} F^k$, with its natural structure of topological vector space, and its natural structure of inner product space. Notice that $F^{(\infty)}$ actually corresponds to the space F^∞ in [9].

When $\mathcal{H}$ is an inner product space, and $n \in \mathbb{N}^*$, $B_n(\mathcal{H}) \subset \mathcal{H}^n$ will denote the space of linearly independent n-tuples of elements of $\mathcal{H}$ (with the convention $B_0(\mathcal{H}) = *$), while $V_n(\mathcal{H}) \subset \mathcal{H}^n$ will denote the space of orthonormal n-tuples of elements of $\mathcal{H}$ (with the convention $V_0(\mathcal{H}) = *$). We also let $B_{\mathcal{H}}$ denote the unit ball of $\mathcal{H}$, $\mathrm{sub}(\mathcal{H})$ denote the set of closed linear subspaces of $\mathcal{H}$, and, for $n \in \mathbb{N}$, $\mathrm{sub}_n(\mathcal{H})$ the set of n-dimensional linear subspaces of $\mathcal{H}$.

When X and Y are compactly-generated spaces, Y^X will denote the space of continuous maps from X to Y, with the k-space topology associated to the compact-open topology. When a group G acts on X and H is a subgroup of G, X^H standardly denotes the subspace of X consisting of the points of X that are fixed by H. We are well aware of the possible conflicts but the context will always help the reader avoid them.

When we have two morphisms $X \to Z$ and $Y \to Z$ of topological spaces, $X \underset{Z}{\times} Y$ will denote the limit of the diagram $X \to Z \leftarrow Y$, and, when no confusion is possible, we will write $X \underset{\Delta}{\times} Y$ instead of $X \underset{Z}{\times} Y$.

If $\mathcal{C}$ is a small category, then:

- $\mathrm{Ob}(\mathcal{C})$ (resp. $\mathrm{Hom}(\mathcal{C})$) will denote its set of objects (resp. of morphisms);

- The structural maps of $\mathcal{C}$ i.e. the initial, final, identity and composition maps are respectively denoted by

$$\mathrm{In}_{\mathcal{C}} : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \quad ; \quad \mathrm{Fin}_{\mathcal{C}} : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C});$$

$$\mathrm{Id}_{\mathcal{C}} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}) \quad \text{and} \quad \mathrm{Comp}_{\mathcal{C}} : \mathrm{Hom}(\mathcal{C}) \underset{\Delta}{\times} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C});$$

- The nerve of $\mathcal{C}$ is denoted by $\mathcal{N}(\mathcal{C})$, whilst $\mathcal{N}(\mathcal{C})_m$ will denote its m-th component for any $m \in \mathbb{N}$.

By a **k-category**, we mean a small category with k-space topologies on the sets of objects and spaces, such that the structural maps induce continuous maps in the category of k-spaces. To every topological category $\mathcal{C}$, we assign a k-category

whose space of objects and space of morphisms are respectively $\mathrm{Ob}(\mathcal{C})_{(k)}$ and $\mathrm{Hom}(\mathcal{C})_{(k)}$.

To a k-category, we may assign its *nerve* in the category of k-spaces, and then take one of the two geometric realizations $\|\ \ \|$ (the "thick realization") or $|\ \ |$ (the "thin realization") of it in the category of k-spaces (see [14]). When $\mathcal{C}$ and $\mathcal{D}$ are two k-categories, we may define another category, denoted by $\mathrm{Func}(\mathcal{C}, \mathcal{D})$, whose objects are the topological functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are the continuous natural transformations between continuous functors from $\mathcal{C}$ to $\mathcal{D}$. The structural maps are obvious. The set $\mathrm{Ob}(\mathrm{Func}(\mathcal{C}, \mathcal{D}))$ is a subset of $\mathrm{Hom}(\mathcal{D})^{\mathrm{Hom}(\mathcal{C})}$, and is given the topology induced by the k-space topology of $\mathrm{Hom}(\mathcal{D})^{\mathrm{Hom}(\mathcal{C})}$. The set $\mathrm{Hom}(\mathrm{Func}(\mathcal{C}, \mathcal{D}))$ is a closed subset of the product $\mathrm{Ob}(\mathrm{Func}(\mathcal{C}, \mathcal{D})) \underset{k}{\times} \mathrm{Ob}(\mathrm{Func}(\mathcal{C}, \mathcal{D})) \underset{k}{\times} \mathrm{Hom}(\mathcal{D})^{\mathrm{Ob}(\mathcal{C})}$ and is equipped with the topology induced by the product topology in the category of k-spaces. From the properties of k-spaces (more precisely the adjunction homeomorphisms), it is easy to check that the structural maps of $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ are continuous, hence $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ is equipped with a structure of k-category.

When M is a topological monoid, we let $\mathcal{B}M$ denote the k-category with one object $*$, such that $\mathrm{Hom}(*, *) = M$ as a topological space, with $x \circ y = x.y$ for any $(x, y) \in M^2$.

Top will denote the category of topological spaces, TopCat the one of topological categories, and kCat the one of k-categories.

Given a topological space X, we will occasionally use X to denote the topological category with X as space of objects *and* space of morphisms and only identity morphisms.

List of important notation

2   Classifying spaces for $\mathbb{V}\mathrm{ect}_G^{F,n}(-)$

In Section 2.1, we introduce the notion of a perfect k-categories, which will be ubiquitous in the rest of the section. In Section 2.2, we start from an n-dimensional vector bundle φ, and use it to construct various categories. These categories are then used to produce, under suitable conditions on φ, a universal n-dimensional G-vector bundle $E \mathrm{Vec}_G^\varphi \to \mathrm{Vec}_G^\varphi$ for G-CW-complexes, when G is a Lie group. Suitable φ's are produced in the end of Section 2.4. In the end, we consider similar results when G-vector bundles are replaced by finite-dimensional G-Hilbert bundles or G-simi-Hilbert bundles.

2.1   Perfect k-categories

Definition 2.1. A non-empty k-category $\mathcal{C}$ is called **perfect** when the structural map

$$\begin{cases} \mathrm{Hom}(\mathcal{C}) & \longrightarrow \mathrm{Ob}(\mathcal{C}) \underset{\mathrm{k}}{\times} \mathrm{Ob}(\mathcal{C}) \\ f & \longmapsto (\mathrm{In}(f), \mathrm{Fin}(f)) \end{cases}$$

is a homeomorphism.

Let X be a k-space. We then let $\mathcal{E}X$ denote the perfect k-category with X as space of objects, $X \underset{\mathrm{k}}{\times} X$ as space of morphisms, and obvious structural maps.

Clearly, $\mathcal{E}X$ is a perfect k-category. This yields an equivalence of categories $\mathcal{E} : \mathrm{CG} \to \mathrm{kCat}$. The following lemmas are easy to check.

Lemma 2.2. *Let $\mathcal{C}$ be a perfect category. Then every full, non-empty, closed subcategory of $\mathcal{C}$ is perfect.*

Lemma 2.3. *Let $\mathcal{C}$ be a perfect category. Then its geometric realization $|\mathcal{C}|$ is contractible.*

Proof. This is proven in the same way as the contractibility of the geometric realization of a (discrete) category which has an initial object. Extra care needs to be taken about the continuity of the involved maps, but this is straightforward. $\square$

Proposition 2.4. *Let $\mathcal{C}$ be a non-empty k-category, and $\mathcal{D}$ a perfect k-category. Then the k-category $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ is perfect.*

Proof. Since $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ is non-empty, we need to prove that the structural map

$$\alpha : \mathrm{Hom}(\mathrm{Func}(\mathcal{C}, \mathcal{D})) \longrightarrow \mathrm{Ob}(\mathrm{Func}(\mathcal{C}, \mathcal{D})) \underset{\mathrm{k}}{\times} \mathrm{Ob}(\mathrm{Func}(\mathcal{C}, \mathcal{D}))$$

is a homeomorphism. Injectivity is straightforward. In order to prove surjectivity, let f and g be two objects in $\mathrm{Func}(\mathcal{C}, \mathcal{D})$. The map $(f \to g) : \begin{cases} \mathrm{Ob}(\mathcal{C}) & \longrightarrow \mathrm{Hom}(\mathcal{D}) \\ x & \longmapsto (f(x), g(x)) \end{cases}$ is continuous, since $\mathcal{D}$ is a perfect k-category. Then $(f, g, f \to g)$ is a morphism from f to g, because all diagrams are commutative in any perfect k-category. It follows that α is a bijection. It thus suffices to prove the continuity of its inverse, and this is a consequence of the continuity of the composite map:

$$\mathrm{Hom}(\mathcal{D})^{\mathrm{Hom}(\mathcal{C})} \underset{\mathrm{k}}{\times} \mathrm{Hom}(\mathcal{D})^{\mathrm{Hom}(\mathcal{C})} \xrightarrow{i \times i} \mathrm{Ob}(\mathcal{D})^{\mathrm{Ob}(\mathcal{C})} \underset{\mathrm{k}}{\times} \mathrm{Ob}(\mathcal{D})^{\mathrm{Ob}(\mathcal{C})} \longrightarrow (\mathrm{Ob}(\mathcal{D}) \underset{\mathrm{k}}{\times} \mathrm{Ob}(\mathcal{D}))^{\mathrm{Ob}(\mathcal{C})},$$

where i is given by right composition by $\mathrm{Id}_{\mathcal{C}}$ and left composition by $\mathrm{In}_{\mathcal{D}}$. The usual properties of k-spaces [16] show that those maps are continuous. $\square$

2.2 Topological categories associated to vector bundles

We fix an integer $n \in \mathbb{N}$ for the rest of the section. Let $\varphi : E \to X$ be an n-dimensional vector bundle over F, and $\tilde{\varphi} : \tilde{E} \to X$ the $\mathrm{GL}_n(F)$-principal bundle canonically associated to it (by considering $\tilde{E}$ as a subspace of $E^{\oplus n}$). We will assume that X is a locally-countable CW-complex. Hence, $\tilde{E}$, E and X are k-spaces, and any finite cartesian product of copies of them also is.

We set
$$\varphi\text{-frame} := \mathcal{E}\tilde{E}.$$

The (right-)action of $\mathrm{GL}_n(F)$ on $\tilde{E}$ induces the diagonal action of $\mathrm{GL}_n(F)$ on $\mathrm{Hom}(\varphi\text{-frame}) = \tilde{E} \times \tilde{E}$. Therefore $\mathrm{GL}_n(F)$ acts on the category φ-frame.

We set
$$\varphi\text{-mod} := \varphi\text{-frame}\,/\mathrm{GL}_n(F),$$
which is obviously a k-category with $\mathrm{Ob}(\varphi\text{-mod}) \cong X$. An object of φ-mod corresponds to a point of X, and a morphism from x to y (with $(x,y) \in X^2$) corresponds to a linear isomorphism $E_x \overset{\cong}{\to} E_y$. The composite of two morphisms $x \overset{f}{\to} y$ and $y \overset{g}{\to} z$ then corresponds to the composite of the corresponding linear isomorphisms $E_x \overset{f}{\to} E_y$ and $E_y \overset{g}{\to} E_z$.

Finally, we define φ-Bdl as the category whose space of objects is E, and whose space of morphisms is the (closed) subspace of $E \underset{\mathrm{k}}{\times} E \underset{\mathrm{k}}{\times} \mathrm{Hom}(\varphi\text{-mod})$ consisting of those triples (e, e', f) such that $\varphi(e) \underset{f}{\longrightarrow} \varphi(e')$ and $f(e) = e'$; the structural maps are obvious. Again, this is clearly a k-category.

Remark 2. When $X = *$ and φ is the canonical vector bundle $F^n \to *$, it is clear from the definitions that φ-frame $\cong \mathcal{E}\mathrm{GL}_n(F)$, and φ-mod $\cong \mathcal{B}\mathrm{GL}_n(F)$.

The definition of φ-mod clearly yields a factor φ-frame $\to \varphi$-mod. On the other hand, we have a functor of k-categories φ-Bdl $\to \varphi$-mod defined as $\varphi : E \to X$ on the spaces of objects and as the third projection $\mathrm{Hom}(\varphi\text{-Bdl}) \to \mathrm{Hom}(\varphi\text{-mod})$ on the spaces of morphisms.

We have a right-action of $\mathrm{GL}_n(F)$ on φ-frame. Identifying F^n with the k-category whose space of objects and space of morphisms are F^n (with identities as the only morphisms), there is a (unique) continuous functor $(\varphi\text{-frame}) \times F^n \to \varphi$-Bdl whose restriction to objects is:

$$\theta : \begin{cases} \tilde{E} \times F^n & \longrightarrow E \\ \big((e_i)_{1 \le i \le n}, (\lambda_i)_{1 \le i \le n}\big) & \longmapsto \sum_{i=1}^{n} \lambda_i . e_i, \end{cases}$$

and which makes the square

$$
\begin{array}{ccc}
\varphi\text{-frame} \times F^n & \longrightarrow & \varphi\text{-Bdl} \\
\downarrow & & \downarrow \\
\varphi\text{-mod} \times F^n & \overset{\pi_1}{\longrightarrow} & \varphi\text{-mod}
\end{array}
$$

commutative (π_1 denotes the projection onto the first factor).

2.3 Universal G-vector bundles

Definition 2.5. Given a Lie group G and an n-dimensional vector bundle φ over the field F, we set

$$\mathrm{Vec}_G^\varphi := \big| \mathrm{Func}(\mathcal{E}G, \varphi\text{-mod}) \big|; \quad \widetilde{\mathrm{Vec}}_G^\varphi := \big| \mathrm{Func}(\mathcal{E}G, \varphi\text{-frame}) \big| \quad \text{and} \quad E\,\mathrm{Vec}_G^\varphi := \big| \mathrm{Func}(\mathcal{E}G, \varphi\text{-Bdl}) \big|.$$

Notice that right-multiplication on the objects of $\mathcal{E}G$ induces a right-action of G on $\mathcal{E}G$. The three functors φ-frame $\longrightarrow$ φ-mod, φ-Bdl $\longrightarrow$ φ-mod and φ-frame $\times F^n \longrightarrow \varphi$-Bdl thus induce, by composition, equivariant continuous functors

$$\mathrm{Func}(\mathcal{E}G, \varphi\text{-frame}) \longrightarrow \mathrm{Func}(\mathcal{E}G, \varphi\text{-mod}), \quad \mathrm{Func}(\mathcal{E}G, \varphi\text{-Bdl}) \longrightarrow \mathrm{Func}(\mathcal{E}G, \varphi\text{-mod}),$$

and

$$\mathrm{Func}(\mathcal{E}G, \varphi\text{-frame}) \times F^n \longrightarrow \mathrm{Func}(\mathcal{E}G, \varphi\text{-Bdl}).$$

Taking geometric realizations everywhere yields three G-maps[2] whose properties are summed up in the next theorem:

Theorem 2.6. *Let X be a locally-countable CW-complex, $\varphi : E \to X$ be an n-dimensional vector bundle over X, and G be a Lie group. Then:*

(i) $\widetilde{\mathrm{Vec}}_G^{\varphi} \longrightarrow \mathrm{Vec}_G^{\varphi}$ is a $(G, GL_n(F))$-principal bundle;

(ii) $E\,\mathrm{Vec}_G^{\varphi} \longrightarrow \mathrm{Vec}_G^{\varphi}$ is an n-dimensional G-vector bundle;

(iii) The canonical map

$$\widetilde{\mathrm{Vec}}_G^{\varphi} \times_{GL_n(F)} F^n \longrightarrow E\,\mathrm{Vec}_G^{\varphi}$$

is an isomorphism of G-vector bundles over Vec_G^{φ}.

Remark 3. When $G = \{1\}$ and $X = *$, we recover the usual universal n-dimensional vector bundle $E\mathrm{GL}_n(F) \times_{\mathrm{GL}_n(F)} F^n \longrightarrow B\mathrm{GL}_n(F)$.

In [9], Theorem 2.6 was claimed to be true with no proof on why the involved maps should be fibre bundles rather than just pseudo-fiber bundles (in there, only the case X is discrete was actually considered). However, the proof of this is long, tedious, and definitely non-trivial. The details however are not necessary to understand the rest of the paper, so we will wait until Section A of the appendix to give a complete proof.

2.4 Classifying spaces for $\mathbb{Vect}_G^{F,n}(-)$

2.4.1 General results

In the upcoming Proposition 2.8, we will see which additional requirements on G and φ are sufficient to ensure that the G-vector bundle $E\,\mathrm{Vec}_G^{\varphi} \longrightarrow \mathrm{Vec}_G^{\varphi}$ is *universal*. Let us dig deeper first into the topology of $\widetilde{\mathrm{Vec}}_G^{\varphi}$. Recall that $\mathrm{GL}_n(F)$ acts freely on $\widetilde{\mathrm{Vec}}_G^{\varphi}$ by a right-action that is compatible with the left-action of G. Thus $G \times \mathrm{GL}_n(F)$ acts on $\widetilde{\mathrm{Vec}}_G^{\varphi}$ by a left-action (of course, $\mathrm{GL}_n(F)$ is considered with the opposite group structure from now on).

---

[2]Local-compactness of G is actually needed here to ensure that the geometric realizations considered here are really G-spaces.

Proposition 2.7. *Let G be a Lie group of dimension m, and $\varphi : E \to X$ be an n-dimensional vector bundle (with underlying field F) such that $\tilde{E}$ is $(m-1)$-connected. Then for every compact subgroup $K \subset G \times GL_n(F)$:*

- *if $K \cap (\{1\} \times GL_n(F)) \neq \{1\}$ then $(\widetilde{\mathrm{Vec}}_G^{\varphi})^K = \emptyset$;*

- *if $K \cap (\{1\} \times GL_n(F)) = \{1\}$, then $(\widetilde{\mathrm{Vec}}_G^{\varphi})^K \simeq *$.*

Proof. Let $K \subset G \times \mathrm{GL}_n(F)$ be a closed subgroup. In the case $K \cap (\{1\} \times \mathrm{GL}_n(F)) \neq \{1\}$, we have $(\widetilde{\mathrm{Vec}}_G^{\varphi})^K = \emptyset$ since $\mathrm{GL}_n(F)$ acts freely on $\widetilde{\mathrm{Vec}}_G^{\varphi}$.

Assume now that $K \cap (\{1\} \times \mathrm{GL}_n(F)) = \{1\}$. First, standard arguments on Lie groups[3] prove that there is a closed subgroup H of G and a continuous group homomorphism $\psi : H \to \mathrm{GL}_n(F)$ such that $K = \{(h, \psi(h)) \mid h \in H\}$.

However $(\widetilde{\mathrm{Vec}}_G^{\varphi})^K = |\mathcal{F}^K|$. By Lemma 2.2, $\mathcal{F}^K$ is either empty or perfect as a full subcategory of the perfect k-category $\mathcal{F}$. We now show that $\mathcal{F}^K$ is non-empty, i.e. we produce a functor $F : \mathcal{E}G \to \varphi$-frame which is invariant by the restriction of the $(G \times \mathrm{GL}_n(F))$-action to K.

On the one hand, we have a left-action of H on $\mathcal{E}G$, by right-multiplication of the inverse. On the other hand, we have a left-action of H on φ-frame, defined by $h.x = x.\psi(h)$ for every $(x, h) \in \tilde{E} \times H$. Then a functor $\mathcal{E}G \to \varphi$-frame is K-invariant if and only if it is H-equivariant. However, an H-equivariant functor is nothing but an H-map $G \to \tilde{E}$ since φ-frame is perfect.

However G has the structure of an H-CW-complex (for the preceding right-action) of dimension $\leq m$ (cf. [6] theorem II). Also, the only isotropy subgroup for the action of H on G is trivial. Finally, the map $\tilde{E} \to *$ induces isomorphisms on the homotopy groups of dimension $i \leq m - 1$. It classically follows that the map $[G, \tilde{E}]_H \longrightarrow [G, *]_H$ induced by $\tilde{E} \to *$ is surjective (use the same line of reasoning as in the proof of Lemma 1.1). Since $[G, *]_H \neq \emptyset$, there is at least one H-map from G to $\tilde{E}$, and we deduce that $\mathcal{F}^K$ is a perfect k-category. The result then follows from Lemma 2.3. $\square$

Proposition 2.8. *Let G be an m-dimensional Lie group, X_1 be a G-CW-complex, and $\varphi : E \to X$ be an n-dimensional vector bundle (with ground field F) such that $\tilde{E}$ is $(m-1)$-connected and X is a locally-countable CW-complex. Pulling back the G-vector bundle $E\,\mathrm{Vec}_G^{\varphi} \to \mathrm{Vec}_G^{\varphi}$ then gives rise to a bijection*

$$[X_1, \mathrm{Vec}_G^{\varphi}]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^{F,n}(X_1).$$

Proof. Let $\varphi_1 : E_1 \to X_1$ be an n-dimensional G-vector bundle over X_1. We let $\tilde{\varphi}_1 : \tilde{E}_1 \to X$ denote the $(G, \mathrm{GL}_n(F))$-principal bundle canonically associated to φ_1. Then $\tilde{E}_1$ is a $(G \times \mathrm{GL}_n(F))$-CW-complex.

---

[3] Define indeed H as the image of K by the canonical projection $\pi_1 : G \times \mathrm{GL}_n(F) \longrightarrow G$; the assumption on K shows that $LK \cap L(G \times \{1\}) = \{0\}$, hence the exponential map yields a neighborhood U of 1_G in G such that $U \cap H$ is closed in U. It follows that H is a closed subgroup of G, hence a Lie group. Thus the restriction of π_1 to K induces a continuous bijection $\alpha : K \xrightarrow{\cong} H$. Since both K and H are Lie groups, α is actually a diffeomorphism, and we may then define ψ as the composite of α^{-1} with the projection $\pi_2 : G \times \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$.

By Proposition 2.7, the G-map $\widetilde{\mathrm{Vec}}_G^{\varphi} \to *$ is a $\mathcal{K}$-weak equivalence, where $\mathcal{K}$ is the class consisting of all closed subgroups K of $G \times \mathrm{GL}_n(F)$ such that $K \cap (\{1\} \times \mathrm{GL}_n(F)) = \{1\}$. Since $\tilde{E}_1$ is a $(G \times \mathrm{GL}_n(F))$-CW-complex, all the isotropy subgroups of which belong to $\mathcal{K}$ (as was shown earlier), we have:

$$[\tilde{E}_1, \widetilde{\mathrm{Vec}}_G^{\varphi}]_{G \times \mathrm{GL}_n(F)} \xrightarrow{\cong} [\tilde{E}_1, *]_{G \times \mathrm{GL}_n(F)} \cong *.$$

Choose $\tilde{f}$ in the unique class of $[\tilde{E}_1, \widetilde{\mathrm{Vec}}_G^{\varphi}]_{G \times \mathrm{GL}_n(F)}$. Then $\tilde{f}$ induces a strong morphism of G-vector bundles

$$
\begin{array}{ccc}
E_1 & \xrightarrow{Ef} & E\,\mathrm{Vec}_G^{\varphi} \\
\varphi_1 \downarrow & & \pi' \downarrow \\
X_1 & \xrightarrow{f} & \mathrm{Vec}_G^{\varphi}.
\end{array}
$$

We deduce that the map $[X_1, \mathrm{Vec}_G^{\varphi}]_G \longrightarrow \mathbb{V}\mathrm{ect}_G^{F,n}(X_1)$ is onto. We will finish by showing that it is one-to-one.

Let $f_1, f_2 : X_1 \to \mathrm{Vec}_G^{\varphi}$ be two maps together with an isomorphism $E_1 = f_1^*(E\,\mathrm{Vec}_G^{\varphi}) \xrightarrow[g]{\cong} f_2^*(E\,\mathrm{Vec}_G^{\varphi}) = E_2$ of G-vector bundles over X_1. Let $\tilde{g} : \tilde{E}_1 \to \tilde{E}_2$ be the associated morphism of $(G, \mathrm{GL}_n(F))$-principal bundles. Then $\tilde{g}$ is a $(G \times \mathrm{GL}_n(F))$-map from $\tilde{E}_1$ to $\tilde{E}_2$. Since $[\tilde{E}_1, \widetilde{\mathrm{Vec}}_G^{\varphi}]_{G \times \mathrm{GL}_n(F)} \cong *$, the maps $\tilde{f}_2 \circ \tilde{g}$ and $\tilde{f}_1$ are $(G \times \mathrm{GL}_n(F))$-homotopic, and it follows that f_1 and f_2 are G-homotopic. $\qquad\square$

2.4.2 Fundamental examples

For any pair $(n, k) \in \mathbb{N}^2$, we let $G_n(F^k)$ denote the space of n-dimensional linear subspaces of F^k, $\gamma_n^k(F) : E_n(F^k) \to G_n(F^k)$ the canonical n-dimensional vector bundle over the Grassmanian manifold $G_n(F^k)$, and $\tilde{\gamma}_n^k(F) : B_n(F^k) \to G_n(F^k)$ the $\mathrm{GL}_n(F)$-principal bundle associated to it. We also let $\gamma_n(F) : E_n(F^{(\infty)}) \to G_n(F^{(\infty)})$ denote the canonical n-dimensional vector bundle over $G_n(F^{(\infty)})$, and $\tilde{\gamma}_n(F)$ the $\mathrm{GL}_n(F)$-principal bundle associated to it. Recall that $G_n(F^k)$ is a finite CW-complex for every pair $(k, n) \in \mathbb{N}^2$, and that $G_n(F^{(\infty)})$, with the topology induced by the filtration $G_n(F^k) \subset G_n(F^{(\infty)})$, is a locally-countable CW-complex for every $n \in \mathbb{N}$. We finally set $\gamma^{(m)}(F) := \coprod_{n \in \mathbb{N}} \gamma_n^{mn}(F)$ for every $m \in \mathbb{N}^*$, and $\gamma(F) := \coprod_{n \in \mathbb{N}} \gamma_n(F)$.

Let G be an N-dimensional Lie group. For every $m \geq \mathbb{N}$, the space $G_k(F^{mk})$ is N-connected for every $k \in \mathbb{N}$ (cf. Theorem 5.1 of [5]). It follows from Proposition 2.8 that, for every $k \in \mathbb{N}$, every $n \geq N$, and every proper G-CW-complex X_1, we have a bijection

$$[X_1, \mathrm{Vec}_G^{\gamma_k^{nk}(F)}]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^{F,k}(X_1)$$

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induced by pulling back the vector bundle $E\operatorname{Vec}_G^{\gamma_k^{nk}(F)} \to \operatorname{Vec}_G^{\gamma_k^{nk}}$. Moreover, for every $k \in \mathbb{N}$, we have a bijection

$$[X_1, \operatorname{Vec}_G^{\gamma_k(F)}]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^{F,k}(X_1).$$

It follows that, for every $k \in \mathbb{N}$, every $n \geq N$, and every connected G-CW-complex X_1, we have bijections

$$\left[X_1, \coprod_{k \in \mathbb{N}} \operatorname{Vec}_G^{\gamma_k^{nk}(F)} \right]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^F(X_1) \quad \text{and} \quad \left[X_1, \coprod_{k \in \mathbb{N}} \operatorname{Vec}_G^{\gamma_k(F)} \right]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^F(X_1).$$

Obviously, these results still hold in the case X_1 is non-connected.

2.5 Similar constructions using isometries or similarities

Here, $F = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.9. A **simi-Hilbert space** is a finite-dimensional vector space V (with ground field F) with a linear family $(\lambda\langle -, - \rangle)_{\lambda \in \mathbb{R}_+^*}$ of inner products on V.

The relevant notion of isomorphisms between two simi-Hilbert spaces is that of similarities. We do have a notion of orthogonality, but no notion of orthonormal families. The relevant notion is that of **simi-orthonormal** families: a family will be said to be simi-orthonormal when it is orthogonal and all its vectors share the same positive norm (for any inner product in the linear family). Equivalently, a family of vectors is simi-orthonormal iff it is orthonormal for some inner product in the linear family. We also have a **simi-orthonormalization process**: if $(e_1, \ldots, e_k)$ is a linearly independent k-tuple in a simi-Hilbert space, there is a unique norm $\| - \|$ in the family such that $\|e_1\| = 1$. We then apply the orthonormalization process to $(e_1, \ldots, e_k)$ with respect to this norm to obtain a simi-orthonormal family. This process is compatible with similarities (i.e. if u is a similarity of V for some inner product space in the family, $(e_1, \ldots, e_k)$ is a linearly independent k-tuple, and $(f_1, \ldots, f_k)$ is obtained from it by the simi-orthonormalization process, then $(u(f_1), \ldots, u(f_k))$ is obtained from $(u(e_1), \ldots, u(e_k))$ by the simi-orthonormalization process), and is continuous with respect to the choice of the family.

Definition 2.10. Let G be a topological group.

- For $n \in \mathbb{N}$, an n-dimensional G-Hilbert bundle is a G-vector bundle with fiber F^n and structural group $U_n(F)$.

- For $n \in \mathbb{N}$, an n-dimensional simi-G-Hilbert bundle is a G-vector bundle with fiber F^n and structural group $\mathrm{GU}_n(F)$.

- A disjoint union of k-dimensional G-Hilbert bundles, for $k \in \mathbb{N}$, is called a G-**Hilbert bundle.**

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- A disjoint union of k-dimensional G-simi-Hilbert bundles, for $k \in \mathbb{N}$, is called a G-**simi-Hilbert bundle.**

Remarks 4. When $G = \{1\}$, we simply speak of Hilbert bundles or simi-Hilbert bundles. Since $U_n(F) \subset \mathrm{GU}_n(F) \subset \mathrm{GL}_n(F)$, any G-Hilbert bundle is in particular a G-simi-Hilbert bundle, and every G-simi-Hilbert bundle is in particular a G-vector bundle.
There is a broader definition of G-Hilbert bundles (see e.g. [12]) which encompasses bundles with infinite-dimensional fibers, but we will not use it.

In a Hilbert bundle, we have a Hilbert space structure on every fiber, and we derive a notion of orthonormal basis on each fiber. In a simi-Hilbert bundle, we only have an inner product on each fiber defined up to a positive scalar, i.e. a structure of simi-Hilbert space on each fiber. From the theory of fibre bundles, we derive familiar notions of (strong) morphisms of G-Hilbert bundles (resp. of G-simi-Hilbert bundles).

If $\varphi : E \to X$ is an n-dimensional G-Hilbert bundle (resp. a G-simi-Hilbert bundle), we may consider the subspace $i\tilde{E} \subset E^{\oplus n}$ (resp. $s\tilde{E} \subset E^{\oplus n}$) consisting of orthonormal bases (resp. of simi-orthonormal bases) on the fibers of φ. The map $i\tilde{E} \to X$ (resp. $s\tilde{E} \to X$) is easily shown to yield a $(G, U_n(F))$-principal bundle (resp. a $(G, \mathrm{GU}_n(F))$-principal bundle), so that the G-fibre bundle with fiber F^n and structural group $U_n(F)$ (resp. $\mathrm{GU}_n(F)$) canonically associated to it is isomorphic to p.

Let X be a G-CW-complex. For every $n \in \mathbb{N}$, we define $i\mathbb{V}\mathrm{ect}_G^{F,n}(X)$ (resp. $s\mathbb{V}\mathrm{ect}_G^{F,n}(X)$) as the set of isomorphism classes of n-dimensional G-Hilbert bundles (resp. G-simi-Hilbert bundles) over X, and $i\mathbb{V}\mathrm{ect}_G^F(X)$ (resp. $s\mathbb{V}\mathrm{ect}_G^F(X)$) as the abelian monoid of isomorphism classes of G-Hilbert bundles (resp. G-simi-Hilbert bundles) over X.

Replacing $\mathrm{GL}_n(F)$-principal bundles by $U_n(F)$-principal bundles (resp. $\mathrm{GU}_n(F)$-principal bundles), and starting from any n-dimensional Hilbert bundle (resp. simi-Hilbert bundle) $\varphi : E \to X$, we may define the k-categories φ-iframe, φ-imod and φ-iBdl (resp. φ-sframe, φ-smod and φ-sBdl), with a construction that is essentially similar to that of φ-frame, φ-mod and φ-Bdl. For every Lie group G, we then obtain G-spaces $i\widetilde{\mathrm{Vec}}_G^\varphi$, $i\mathrm{Vec}_G^\varphi$ and $Ei\mathrm{Vec}_G^\varphi$ (resp. $s\widetilde{\mathrm{Vec}}_G^\varphi$, $s\mathrm{Vec}_G^\varphi$ and $Es\mathrm{Vec}_G^\varphi$).

We may then prove results that are similar to Theorem 2.6 and to Proposition 2.8. To be more precise, on the one hand, if $\varphi : E \to X$ is an n-dimensional Hilbert bundle such that X is a locally-countable CW-complex, then $i\widetilde{\mathrm{Vec}}_G^\varphi \to i\mathrm{Vec}_G^\varphi$ has a structure of $(G, U_n(F))$ principal bundle, $Ei\mathrm{Vec}_G^\varphi \to i\mathrm{Vec}_G^\varphi$ a structure of G-Hilbert bundle, and the natural map $i\widetilde{\mathrm{Vec}}_G^\varphi \times_{U_n(F)} F^n \to Ei\mathrm{Vec}_G^\varphi$ is an isomorphism of Hilbert bundles. On the other hand, if $\varphi : E \to X$ is an n-dimensional simi-Hilbert bundle such that X is a locally-countable CW-complex, then $s\widetilde{\mathrm{Vec}}_G^\varphi \to s\mathrm{Vec}_G^\varphi$ has a structure of $(G, \mathrm{GU}_n(F))$-principal bundle, $Es\mathrm{Vec}_G^\varphi \to s\mathrm{Vec}_G^\varphi$ a structure of G-simi-Hilbert bundle, and the natural map $s\widetilde{\mathrm{Vec}}_G^\varphi \times_{\mathrm{GU}_n(F)} F^n \to Es\mathrm{Vec}_G^\varphi$ is an isomorphism of G-simi-Hilbert bundles.

Moreover, if $\varphi : E \to X$ is an n-dimensional Hilbert bundle such that $\tilde{E}$ is $(m-1)$-connected with $m = \dim G$, the map $[X_1, i\operatorname{Vec}_G^\varphi]_G \to i\mathbb{V}\mathrm{ect}_G^{F,n}(X_1)$, induced by pulling back the universal G-Hilbert bundle $Ei\operatorname{Vec}_G^\varphi \to i\operatorname{Vec}_G^\varphi$, is a bijection for every G-CW-complex X_1. Finally, if $\varphi : E \to X$ is an n-dimensional simi-Hilbert bundle such that $\tilde{E}$ is $(m-1)$-connected with $m = \dim G$, then the map $[X_1, s\operatorname{Vec}_G^\varphi]_G \to s\mathbb{V}\mathrm{ect}_G^{F,n}(X_1)$, induced by pulling back the G-simi-Hilbert bundle $Es\operatorname{Vec}_G^\varphi \to s\operatorname{Vec}_G^\varphi$, is a bijection for every G-CW-complex X_1.

We finish this section with an easy result that establishes a relationship between the three constructions. If $\varphi : E \to X$ is an n-dimensional Hilbert bundle, then the inclusion $i\tilde{E} \subset s\tilde{E}$ induces a canonical functor $\varphi\text{-iframe} \to \varphi\text{-sframe}$. If $\varphi : E \to X$ is an n-dimensional simi-Hilbert bundle, then $s\tilde{E} \subset \tilde{E}$ induces a functor $\varphi\text{-sframe} \to \varphi\text{-frame}$. All those functors induce G-maps between the G-spaces $\operatorname{Vec}_G^\varphi$, $E\operatorname{Vec}_G^\varphi$, etc. . . that were previously defined using φ. The next proposition, the proof of which is straightforward, sums up their properties:

Proposition 2.11.

(a) *Let $\varphi : E \to X$ be an n-dimensional Hilbert bundle, and G be a Lie group. Then the canonical diagram*

$$
\begin{array}{ccc}
Ei\operatorname{Vec}_G^\varphi & \longrightarrow & Es\operatorname{Vec}_G^\varphi \\
\downarrow & & \downarrow \\
i\operatorname{Vec}_G^\varphi & \longrightarrow & s\operatorname{Vec}_G^\varphi
\end{array}
$$

defines a strong morphism of G-simi-Hilbert bundles.

(b) *Let $\varphi : E \to X$ be an n-dimensional simi-Hilbert bundle, and G be a Lie group. Then, the canonical diagram*

$$
\begin{array}{ccc}
Es\operatorname{Vec}_G^\varphi & \longrightarrow & E\operatorname{Vec}_G^\varphi \\
\downarrow & & \downarrow \\
s\operatorname{Vec}_G^\varphi & \longrightarrow & \operatorname{Vec}_G^\varphi
\end{array}
$$

defines a strong morphism of G-vector bundles.

3   A construction of equivariant Γ-spaces

In this section, $F = \mathbb{R}$ or $\mathbb{C}$. Our goal here is to construct a $\Gamma - G$-space $\underline{A}$ such that $\underline{A}(\mathbf{1})$ is a classifying space for the monoid-valued functor $\mathbb{V}\mathrm{ect}_G^F(-)$ on the category of proper G-CW-complexes. In order to achieve this, we introduce pre-decomposed G-Hilbert bundles, i.e. families of Hilbert bundles (over the same base space) indexed over a finite set: they will be defined in Section 3.1 as the objects of a certain category $\Gamma\text{-Fib}^F$. In Sections 3.2 and 3.3, we extend the -mod, -frame, -Bdl, etc. . . constructions from Hilbert bundles with

fixed dimension to pre-decomposed Hilbert bundles, and then give basic results about those. In Section 3.4, we define so-called "Hilbert Γ-bundles", a certain type of contravariant functors from Γ to Γ-Fib^F. Any such functor will induce, after composition with the -mod construction, a functor from Γ to kCat, and therefore, after composition with $|\operatorname{Func}(\mathcal{E}G, -)|$, we will finally recover an equivariant Γ-space. In Section 3.4.3, we produce particular Hilbert Γ-bundles Fib^{F^m} for each $m \in \mathbb{N}^* \cup \{\infty\}$ and prove that the equivariant Γ-spaces $\underline{\operatorname{Vec}}_G^{F,m}(-)$ derived from them are such that $\underline{\operatorname{Vec}}_G^{F,m}(\mathbf{1})$ is homotopy equivalent to $\operatorname{Vec}_G^{\gamma^{(m)}(F)}$. We deduce that $\underline{\operatorname{Vec}}_G^{F,m}(\mathbf{1})$ is a classifying space for $\mathbb{V}\mathrm{ect}_G^F(-)$ on the category of proper G-CW-complexes whenever $m \geq \dim(G)$, and that $\underline{\operatorname{Vec}}_G^{F,m}(\mathbf{1})$, with a suitable H-space structure, classifies $\mathbb{V}\mathrm{ect}_G^F(-)$ as a monoid-valued functor. From Section 3.1 to 3.4.3, we systematically make the parallel constructions involving isometries and similarities, just like in Section 2.5. In Section 3.4.4, the relationship between those three classes of Γ-spaces is investigated upon. Finally, we will show in Section 3.4.5 that natural transformations of Hilbert Γ-bundles induce homotopies between associated Γ-spaces, and apply this to some obvious transformations between the Hilbert Γ-bundles Fib^{F^m} for $m \in \mathbb{N}^* \cup \{\infty\}$.

3.1 The category Γ-Fib$_F$

3.1.1 Definition

We define the category Γ-Fib$_F$ as follows:

- An object of Γ-Fib$_F$ consists of a finite set S, a locally-countable CW-complex X, and, for every $s \in S$, of a Hilbert bundle $p_s : E_s \to X$ with ground field F. Such an object is called an **S-object** over X. If $S = \mathbf{n}$ for some $n \in \mathbb{N}$, an S-object will be called an n-object.

- A morphism $f : (S, X, (p_s)_{s \in S}) \longrightarrow (T, Y, (q_t)_{t \in T})$ consists of a morphism $\gamma : T \to S$ in the category Γ, a continuous map $\bar{f} : X \to Y$, and, for every $t \in T$, a strong morphism of Hilbert bundles

$$
\begin{array}{ccc}
\underset{s \in \gamma(t)}{\oplus} E_s & \xrightarrow{\ f_t\ } & E'_t \\
{\scriptstyle \underset{s \in \gamma(t)}{\oplus} p_s} \downarrow & & \downarrow {\scriptstyle q_t} \\
X & \xrightarrow[\ \bar{f}\ ]{} & Y.
\end{array}
$$

If $f : (S, X, (p_s)_{s \in S}) \to (T, Y, (q_t)_{t \in T})$ is the morphism in Γ-Fib$_F$ corresponding to $(\gamma, \bar{f}, (f_t)_{t \in T})$, and $g : (T, Y, (q_t)_{t \in T}) \to (U, Z, (r_u)_{u \in U})$ is the morphism in Γ-Fib$_F$ corresponding to $(\gamma', \bar{g}, (g_u)_{u \in U})$, then the composite morphism $g \circ f : (S, X, (p_s)_{s \in S}) \to (U, Z, (r_u)_{u \in U})$ is the one which corresponds to the triple consisting of $\gamma \circ \gamma'$, $\bar{g} \circ \bar{f}$, and the family $\left(g_u \circ \left[\underset{t \in \gamma'(u)}{\oplus} f_t \right] \right)_{u \in U}$.

3.1.2 The forgetful functor $\mathcal{O}_\Gamma^F : \Gamma\text{-}\mathbf{Fib}_F \longrightarrow \Gamma$

$$\mathcal{O}_\Gamma^F : \begin{cases} \Gamma\text{-}\mathrm{Fib}_F & \longrightarrow \Gamma \\ (S, X, (p_s)_{s\in S}) & \longmapsto S \\ f = (\gamma, \bar{f}, (f_t)_{t\in T}) : (S, X, (p_s)_{s\in S}) \to (T, Y, (q_t)_{t\in T}) & \longmapsto (\gamma : T \to S) \end{cases}$$

is a contravariant functor from $\Gamma\text{-}\mathrm{Fib}_F$ to Γ.

3.1.3 Sums in the category $\Gamma\text{-}\mathbf{Fib}_F$

Given a finite set T, and, for every $t \in T$, an object $x_t = (S_t, X_t, (p_s^t)_{s\in S_t})$ of $\Gamma\text{-}\mathrm{Fib}_F$, we define:

$$\sum_{t\in T} x_t := \left(\coprod_{t\in T} S_t, \prod_{t\in T} X_t, \left(p_s^t \times \prod_{t_1 \in T\smallsetminus\{t\}} \mathrm{id}_{X_{t_1}} \right)_{t\in T, s\in S_t} \right).$$

In particular, if p is a 1-object, then $n.p$ is an n-object (with $\coprod_{i=1}^{n} \{1\} = \{1, \dots, n\}$).

For example, the sum of two 1-objects $(X, p : E \to X)$ and $(Y, q : E' \to Y)$ of $\Gamma\text{-}\mathrm{Fib}_F$ is $(\{1, 2\}, X \times Y, (p_1, p_2))$, where

$$
\begin{array}{ccc}
E \times Y & \longrightarrow & E \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\
X \times Y & \underset{\pi_1}{\longrightarrow} & X
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X \times E' & \longrightarrow & E' \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q} \\
X \times Y & \underset{\pi_2}{\longrightarrow} & Y
\end{array}
$$

are pull-back squares (π_1 and π_2 respectively denote the canonical projections).

3.2 Various functors from $\Gamma\text{-}\mathbf{Fib}_F$ to kCat

We wish to extend the constructions φ-mod and φ-Bdl from Section 2 to functors $\Gamma\text{-}\mathrm{Fib}_F \longrightarrow \mathrm{kCat}$.

3.2.1 The dimension over a 1-object

Let $(X, p : E \to X)$ be a 1-object of $\Gamma\text{-}\mathrm{Fib}_F$. The natural map $\dim_p : \begin{cases} X & \longrightarrow \mathbb{N} \\ x & \longmapsto \dim(E_x) \end{cases}$

is continuous because X is a CW-complex. Thus, setting $X_n := \dim_p^{-1}\{n\}$, $E_n := p^{-1}(X_n)$, and $p_n = p_{|E_n} : E_n \to X_n$, then $p = \coprod_{n\in\mathbb{N}} p_n$.

3.2.2 The functor -mod

For any 1-object p over X, i.e. any a Hilbert bundle, we define

$$p\text{-mod} := \coprod_{n \in \mathbb{N}} (p_n\text{-mod}).$$

We then obtain a canonical functor p-mod $\to \mathcal{E}X$ by identifying the spaces of objects. For any S-object $\varphi = (S, X, (p_s : E_s \to X)_{s \in S})$, we define the category φ-mod as follows: an object of φ-mod is a point $x \in X$, and a morphism $x \to y$ in φ-mod is a family $(\varphi_s)_{s \in S}$ of linear isomorphisms $\varphi_s : (E_s)_x \xrightarrow{\cong} (E_s)_y$. As a topological category, φ-mod is defined as the fiber product of the categories p_s-mod over $\mathcal{E}X$ for all $s \in S$.

Let $f : \varphi = (S, X, (p_s)_{s \in S}) \to (T, Y, (q_t)_{t \in T})$ be a morphism in Γ-Fib$_F$, with corresponding morphisms $\gamma : T \to S$, $\bar{f} : X \to Y$ and $(f_t)_{t \in T}$. We assign a morphism f-mod to f as follows:

- For every object $x \in X$ of φ-mod, $(f\text{-mod})(x) := \bar{f}(x)$;

- Let $(\varphi_s)_{s \in S}$ be a morphism in φ-mod. For every $t \in T$, we set

$$\psi_t := f_t \circ \left(\bigoplus_{s \in \gamma(t)} \varphi_s \right) \circ (f_t)^{-1}$$

  so that the squares

$$
\begin{array}{ccc}
\bigoplus_{s \in \gamma(t)} (E_s)_x & \xrightarrow{(f_t)_x} & (E_t)_{f(x)} \\
{\scriptstyle \bigoplus_{s \in \gamma(t)} \varphi_s} \downarrow & & \downarrow {\scriptstyle \psi_t} \\
\bigoplus_{s \in \gamma(t)} (E_s)_y & \xrightarrow{(f_t)_y} & (E_t)_{f(y)}
\end{array}
$$

  are all commutative. We then define

$$(f\text{-mod})((\varphi_s)_{s \in S}) := (\psi_t)_{t \in T}.$$

It is easily checked that this definition is compatible with the composition of morphisms, and it thus yields a functor:

$$\text{-mod} : \Gamma\text{-Fib}_F \longrightarrow \text{kCat}.$$

3.2.3 The functors -imod and -smod

For every 1-object p, we set

$$p\text{-imod} := \coprod_{n \in \mathbb{N}} (p_n\text{-imod}).$$

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By a construction that is strictly similar to that of -mod, we then recover a functor:

$$\text{-imod} : \Gamma\text{-Fib}_F \longrightarrow \text{kCat}.$$

For any S-object $\varphi = (S, X, (p_s : E_s \to X)_{s \in S})$, an object of φ-imod simply corresponds to a point $x \in X$, while a morphism $x \to y$ in φ-imod is a family $(\varphi_s)_{s \in S}$ of unitary morphisms $\varphi_s : (E_s)_x \overset{\cong}{\to} (E_s)_y$.

Let $p : E \to X$ be a 1-object over X. We set

$$p\text{-smod} := \coprod_{n \in \mathbb{N}} (p_n\text{-smod}).$$

We obtain a functor p-smod $\to \mathcal{E}X \times \mathcal{B}\mathbb{R}_+^*$ by assigning $(x, y, \|\varphi\|)$ to every morphism $\varphi : E_x \to E_y$ (here, $\|\varphi\|$ denotes the norm of the similarity φ with respect to the respective inner product structures on E_x and E_y): this is compatible with the composition of morphisms, since we are dealing with similarities here.

For any S-object $\varphi = (S, X, (p_s)_{s \in S})$, φ-smod is defined as the fiber product of the categories p_s-smod over $\mathcal{E}X \times \mathcal{B}\mathbb{R}_+^*$ for all $s \in S$.

For any S-object $\varphi = (S, X, (p_s : E_s \to X)_{s \in S})$, an object of φ-smod simply corresponds to a point $x \in X$, while a morphism $x \to y$ in φ-smod is a family $(\varphi_s)_{s \in S}$ of similarities $\varphi_s : (E_s)_x \overset{\cong}{\to} (E_s)_y$ *which share the same norm*. It is then easy to extend this construction to obtain a functor

$$\text{-smod} : \Gamma\text{-Fib}_F \longrightarrow \text{kCat}.$$

The key point here is that the orthogonal direct sum of similarities is not necessarily a similarity. The condition on the objects of φ-smod that all similarities in the family share the same norm ensures that any orthogonal direct sum of them is a similarity.

3.2.4 The functor -Bdl

Let p be a 1-object over X.

We set

$$p\text{-Bdl} := \coprod_{n \in \mathbb{N}} (p_n\text{-Bdl}).$$

As in the case of p-mod, we have a canonical functor p-Bdl $\to \mathcal{E}X$.

For every S-object $\varphi = (S, X, (p_s : E_s \to X)_{s \in S})$, φ-Bdl is defined as the fiber product of the categories p_s-Bdl over $\mathcal{E}X$ for all $s \in S$. An object of φ-Bdl simply corresponds to a family $(e_s)_{s \in S}$ such that, for some $x \in X$: $\forall s \in S$, $e_s \in (E_s)_x$. A morphism $(e_s)_{s \in S} \to (e'_s)_{s \in S}$ in φ-Bdl, with $e_s \in (E_s)_x$ and $e'_s \in (E_s)_y$ for all $s \in S$, is a family $(\varphi_s)_{s \in S}$ of linear isomorphisms $\varphi_s : (E_s)_x \overset{\cong}{\to} (E_s)_y$ such that $\varphi_s(e_s) = e'_s$ for all $s \in S$.

As in the case of -mod, we can extend -Bdl to obtain a functor

$$\text{-Bdl} : \Gamma\text{-Fib}_F \longrightarrow \text{kCat}.$$

The functors φ-Bdl $\to$ φ-mod of Section 2 then induce a natural transformation:

$$\text{-Bdl} \longrightarrow \text{-mod}.$$

3.2.5 The functors -iBdl and -sBdl

For any 1-object p, we set

$$p\text{-iBdl} := \coprod_{n \in \mathbb{N}} (p_n\text{-iBdl}).$$

As in the case of -Bdl, we recover a functor -iBdl : Γ-Fib$_F \longrightarrow$ kCat together with a natural transformation -iBdl $\longrightarrow$ -imod.

For any 1-object p over X, we set

$$p\text{-sBdl} := \coprod_{n \in \mathbb{N}} (p_n\text{-sBdl}).$$

This time, we do not form the fiber product over $\mathcal{E}X$, rather over $\mathcal{E}X \times \mathcal{B}\mathbb{R}_+^*$, as in the construction of -smod. We obtain a functor -sBdl : Γ-Fib$_F \longrightarrow$ kCat together with a natural transformation -sBdl $\longrightarrow$ -smod.

3.3 Fundamental results on the previous functors

Proposition 3.1. *Let $p : E \to X$ be a 1-object and $n \in \mathbb{N}^*$. Then*

(i) $(p\text{-mod})^n \cong (n.p)\text{-mod}$.

(ii) $(p\text{-imod})^n \cong (n.p)\text{-imod}$.

(iii) *There are three continuous functors $F^p : (n.p)\text{-smod} \to (p\text{-smod})^n$, $G^p : (p\text{-smod})^n \to (n.p)\text{-smod}$ and $H^p : (p\text{-smod})^n \times I \longrightarrow (p\text{-smod})^n$ such that*

$$G^p \circ F^p = \mathrm{id}_{(n.p)\text{-smod}}, \; H^p_{|(p\text{-smod})^n \times \{0\}} = \mathrm{id}_{(p\text{-smod})^n} \; \text{and} \; H^p_{|(p\text{-smod})^n \times \{1\}} = F^p \circ G^p.$$

Remark 5. There are similar results for the functors -Bdl, -iBdl and -sBdl.

Proof. The space of objects of $(n.p)$-mod is X^n, and this is precisely the space of objects of $(p\text{-mod})^n$. A morphism in $(n.p)$-mod is defined by two objects $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ and an n-tuple of linear isomorphisms $\varphi_i : E_{x_i} \xrightarrow{\sim} E_{y_i}$ (for $i \in \{1, \ldots, n\}$); this corresponds naturally to a morphism in $(p\text{-mod})^n$. We thus have a canonical isomorphism between the categories $(n.p)$-mod and $(p\text{-mod})^n$. Similarly, we have a canonical isomorphism between the categories $(n.p)$-imod and $(p\text{-imod})^n$. However, this fails for similarities, because of the norm condition in the definition of the morphisms of p-smod.

We define $F^p : (n.p)\text{-smod} \longrightarrow (p\text{-smod})^n$ by assigning the n-tuple of morphisms $((x_1, y_1, \varphi_1), \ldots, (x_n, y_n, \varphi_n))$ of p-smod to every morphism $((x_1, \ldots, x_n), (y_1, \ldots, y_n), (\varphi_1, \ldots, \varphi_n))$ of $(n.p)$-smod.

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We define $G^p : (p\text{-smod})^n \longrightarrow (n.p)\text{-smod}$ by assigning the morphism $\left((x_1, \ldots, x_n), (y_1, \ldots, y_n), \left(\varphi_1, \frac{\|\varphi_1\|}{\|\varphi_2\|} \cdot \varphi_2, \ldots, \frac{\|\varphi_1\|}{\|\varphi_n\|} \cdot \varphi_n\right)\right)$ of $(n.p)$-smod to every n-tuple $((x_1, y_1, \varphi_1), \ldots, (x_n, y_n, \varphi_n))$ of morphisms of p-smod. We readily see that $G^p \circ F^p = \mathrm{id}_{n.p\text{-smod}}$.

Finally, we define $H^p : (p\text{-smod})^n \times I \longrightarrow (p\text{-smod})^n$ by assigning the morphism $\left((x_1, y_1, \varphi_1), \left(x_2, y_2, \left(\frac{\|\varphi_1\|}{\|\varphi_2\|}\right)^t .\varphi_2\right), \ldots, \left(x_n, y_n, \left(\frac{\|\varphi_1\|}{\|\varphi_n\|}\right)^t .\varphi_n\right)\right)$ of $(p\text{-smod})^n$ to every morphism $((x_1, y_1, \varphi_1), \ldots, (x_n, y_n, \varphi_n),)$ of $(p\text{-smod})^n \times I$. It is then easy to check that H^p is a functor, that its restriction to $(p\text{-smod})^n \times \{0\}$ is $\mathrm{id}_{(p\text{-smod})^n}$ and that its restriction to $(p\text{-smod})^n \times \{1\}$ is $F^p \circ G^p$. $\qquad\square$

Proposition 3.2. *Let p and q be two Hilbert bundles together with two strong morphisms of Hilbert bundles*

$$
\begin{array}{ccc}
E \xrightarrow{\ \tilde{f}\ } E' & & E' \xrightarrow{\ \tilde{g}\ } E \\
\ \downarrow{\scriptstyle p} \qquad \downarrow{\scriptstyle q} & and & \ \downarrow{\scriptstyle p} \qquad \downarrow{\scriptstyle q} \\
X \xrightarrow[\ f\ ]{} Y & & Y \xrightarrow[\ g\ ]{} X
\end{array} .
$$

Then these squares respectively induce a functor $f\text{-mod} : p\text{-mod} \to q\text{-mod}$ and a functor $g\text{-mod} : q\text{-mod} \to p\text{-mod}$, together with two natural transformations

$$\eta : \mathrm{id}_{p\text{-mod}} \to (g\text{-mod}) \circ (f\text{-mod}) \quad and \quad \varepsilon : \mathrm{id}_{q\text{-mod}} \to (f\text{-mod}) \circ (g\text{-mod}).$$

Similar results hold for the functors -imod and -smod.

Corollary 3.3. *Let S be a finite set, φ and ψ be two S-objects of $\Gamma\text{-Fib}_F$, and $f : \varphi \to \psi$ and $g : \psi \to \varphi$ be two morphisms such that $\mathcal{O}_\Gamma^F(f) = \mathcal{O}_\Gamma^F(g) = \mathrm{id}_S$. Then there are two natural transformations*

$$\eta : \mathrm{id}_{\varphi\text{-mod}} \to (g\text{-mod}) \circ (f\text{-mod}) \quad and \quad \varepsilon : \mathrm{id}_{\psi\text{-mod}} \to (f\text{-mod}) \circ (g\text{-mod}).$$

Similar results hold for the functors -imod and -smod.

Proof of Proposition 3.2: In order to build η, it suffices to construct, for every $x \in X$, an isomorphism between E_x et $E_{g(f(x))}$ (and to do this in a continuous way, of course). This isomorphism is simply given by: $\begin{cases} E_x & \to E_{g(f(x))} \\ y & \mapsto \tilde{g}(\tilde{f}(y)). \end{cases}$ The construction of ε is similar. $\qquad\square$

3.4 Hilbert Γ-bundles

3.4.1 Definition

Definition 3.4. A **Hilbert Γ-bundle** is a contravariant functor $\varphi : \Gamma \longrightarrow \Gamma\text{-Fib}_F$ which satisfies the following conditions:

(i) $\mathcal{O}_\Gamma^F \circ \varphi = \mathrm{id}_\Gamma$;

(ii) $\varphi(\mathbf{0}) = (\mathbf{0}, *, \emptyset)$;

(iii) For every $n \in \mathbb{N}^*$, there exists a morphism $f_n : n.\varphi(\mathbf{1}) \to \varphi(\mathbf{n})$ in $\Gamma\text{-Fib}_F$ such that $\mathcal{O}_\Gamma^F(f_n) = \mathrm{id}_{\mathbf{n}}$.

Remark 6. Notice that in condition (iii), only the existence of some morphisms is required, but they are not part of the structure of a Hilbert Γ-bundle.

3.4.2 Induced constructions

Definition 3.5. Let φ be an object of $\Gamma - \mathrm{Fib}_F$ and G be a Lie group. We define

$$\mathrm{Vec}_G^\varphi := |\mathrm{Func}(\mathcal{E}G, \varphi\text{-mod})|, \quad i\,\mathrm{Vec}_G^\varphi := |\mathrm{Func}(\mathcal{E}G, \varphi\text{-imod})|, \quad \text{and} \quad s\,\mathrm{Vec}_G^\varphi := |\mathrm{Func}(\mathcal{E}G, \varphi\text{-smod})|.$$

In what follows, we let φ be a Hilbert Γ-bundle and G be a Lie group.

Definition 3.6. We define three functors from Γ to CG_G:

$$\underline{\mathrm{Vec}}_G^\varphi : S \longmapsto \mathrm{Vec}_G^{\varphi(S)}, \quad i\underline{\mathrm{Vec}}_G^\varphi : S \longmapsto i\,\mathrm{Vec}_G^{\varphi(S)} \quad \text{and} \quad s\underline{\mathrm{Vec}}_G^\varphi : S \longmapsto s\,\mathrm{Vec}_G^{\varphi(S)}.$$

Proposition 3.7. *The three functors* $\underline{\mathrm{Vec}}_G^\varphi$, $i\underline{\mathrm{Vec}}_G^\varphi$ *and* $s\underline{\mathrm{Vec}}_G^\varphi$ *define* $\Gamma - G$-*spaces.*

Proof. We only prove the proposition for $\underline{\mathrm{Vec}}_G^\varphi$ and $s\underline{\mathrm{Vec}}_G^\varphi$, since the proof for $i\underline{\mathrm{Vec}}_G^\varphi$ is similar to that for $\underline{\mathrm{Vec}}_G^\varphi$.

• **The case of** Vec_G^φ. We first notice that $\varphi(\mathbf{0})\text{-mod} = *$, and so $\underline{\mathrm{Vec}}_G^\varphi(\mathbf{0}) = *$.

The maps from $\mathbf{1}$ to $\mathbf{n}$ induce a morphism $g_n : \varphi(\mathbf{n}) \to \varphi(\mathbf{1})^n$ such that $\mathcal{O}_\Gamma^F(g_n) = \mathrm{id}_{\mathbf{n}}$. By assumption on φ, using Proposition 3.1 shows there is also a morphism $f_n : \varphi(\mathbf{1})^n \longrightarrow \varphi(\mathbf{n})$ such that $\mathcal{O}_\Gamma^F(f_n) = \mathrm{id}_{\mathbf{n}}$. Using Corollary 3.3, we deduce that f_n and g_n respectively induce equivariant continuous functors

$$\tilde{f}_n : \mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})^n\text{-mod}) \longrightarrow \mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-mod})$$

and

$$\tilde{g}_n : \mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-mod}) \longrightarrow \mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})^n\text{-mod}),$$

and equivariant natural transformations $\tilde{\eta} : \mathrm{id} \longrightarrow \tilde{f}_n \circ \tilde{g}_n$ and $\tilde{\eta} : \mathrm{id} \longrightarrow \tilde{g}_n \circ \tilde{f}_n$. Therefore, g_n induces an equivariant homotopy equivalence

$$|\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-mod})| \overset{\simeq}{\longrightarrow} |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})^n\text{-mod})|.$$

From Proposition 3.1, we derive that $(\varphi(\mathbf{1})\text{-mod})^n \cong \varphi(\mathbf{1})^n\text{-mod}$, and it follows that

$$|\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})^n\text{-mod})| \simeq |\mathrm{Func}(\mathcal{E}G, (\varphi(\mathbf{1})\text{-mod})^n)| \cong |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})\text{-mod})|^n.$$

We thus have an equivariant homotopy equivalence

$$\mathrm{Vec}_G^\varphi(\mathbf{n}) = |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-mod})| \xrightarrow{\simeq} |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})\text{-mod})|^n = \mathrm{Vec}_G^\varphi(\mathbf{1})^n,$$

and it is easy to check that it is induced by the maps from $\mathbf{1}$ to $\mathbf{n}$. We conclude that $\underline{\mathrm{Vec}}_G^\varphi$ is a $\Gamma - G$-space.

• **The case of** $s\,\mathrm{Vec}_G^\varphi$**.** As in the above case, we obtain an equivariant homotopy equivalence

$$|\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-smod})| \xrightarrow{\simeq} |\mathrm{Func}(\mathcal{E}G, [n.\varphi(\mathbf{1})]\text{-smod})|.$$

We set $p := \varphi(\mathbf{1})$. With the notations from Proposition 3.1, we obtain functors $F^p : (n.p)\text{-smod} \to (p\text{-smod})^n$, $G^p : (p\text{-smod})^n \to (n.p)\text{-smod}$ and $H^p : (p\text{-smod})^n \times I \longrightarrow (p\text{-smod})^n$.

We thus recover three equivariant functors:

$$\tilde{F}^p : \mathrm{Func}(\mathcal{E}G, (n.p)\text{-smod}) \longrightarrow \mathrm{Func}(\mathcal{E}G, (p\text{-smod})^n),$$

$$\tilde{G}^p : \mathrm{Func}(\mathcal{E}G, (p\text{-smod})^n) \times I \longrightarrow \mathrm{Func}(\mathcal{E}G, n.p\text{-smod}),$$

and

$$\tilde{H}^p : \mathrm{Func}(\mathcal{E}G, (p-\mathrm{smod})^n) \times I \longrightarrow \mathrm{Func}(\mathcal{E}G, (p-\mathrm{smod})^n).$$

We easily find that $\tilde{G}^p \circ \tilde{F}^p = \mathrm{id}$, and that $\tilde{H}^p$ is an equivariant homotopy from id to $\tilde{F}^p \circ \tilde{G}^p$. After taking the geometric realizations, we obtain that $|\tilde{F}^p|$ is an equivariant homotopy equivalence and $|\tilde{G}^p|$ is an equivariant homotopy inverse of $|\tilde{F}^p|$. Therefore, the natural map $|\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-smod})| \xrightarrow{\simeq} |\mathrm{Func}(\mathcal{E}G, (\varphi(\mathbf{1})\text{-smod})^n)|$ is an equivariant homotopy equivalence. However, we know that $|\mathrm{Func}(\mathcal{E}G, (\varphi(\mathbf{1})\text{-smod})^n)| \cong |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})\text{-smod})|^n$. Therefore, the map

$$s\,\mathrm{Vec}_G^\varphi(\mathbf{n}) = |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{n})\text{-smod})| \xrightarrow{\simeq} |\mathrm{Func}(\mathcal{E}G, \varphi(\mathbf{1})\text{-smod})|^n = (s\,\mathrm{Vec}_G^\varphi(\mathbf{1}))^n,$$

induced by the maps from $\mathbf{1}$ to $\mathbf{n}$, is an equivariant homotopy equivalence. We conclude that $s\underline{\mathrm{Vec}}_G^\varphi$ is a $\Gamma - G$-space. $\qquad\square$

3.4.3 Fundamental examples of Hilbert Γ-bundles

Definition 3.8. For every finite set S, we let $\Gamma(S)$ denote the set of maps $f : \mathcal{P}(S) \to \mathcal{P}(\mathbb{N})$ which respect disjoint unions (and in particular $f(\emptyset) = \emptyset$), and such that $f(S)$ is finite. We will write $S \xrightarrow{f} \mathbb{N}$ when $f \in \Gamma(S)$.

For an inner product space $\mathcal{H}$ (with underlying field F) of finite dimension or isomorphic to $F^{(\infty)}$, and for a finite subset A of $\mathbb{N}$, we may consider the inner product space $\mathcal{H}^A$ as embedded in the Hilbert space $\mathcal{H}^\infty$. We then define $G_A(\mathcal{H})$ as the set of subspaces of dimension $\#A$ of $\mathcal{H}^A$, with the limit topology for the inclusion of $G_{\#A}(E)$, where E ranges over the finite dimensional subspaces of $\mathcal{H}$. When A is empty, we set $G_\emptyset(\mathcal{H}) = *$.

We let $p_A(\mathcal{H}) : E_A(\mathcal{H}) \longrightarrow G_A(\mathcal{H})$ denote the canonical Hilbert bundle of dimension $\#A$ over $G_A(\mathcal{H})$. The set $E_A(\mathcal{H})$ is constructed as a subspace of the product of $\mathcal{H}^A$ (with the limit topology described above) with $G_A(\mathcal{H})$.

Remark 7. In any case, $G_A(\mathcal{H})$ is a countable CW-complex.

For every finite set S, we define the following object of Γ-Fib_F:

$$\mathrm{Fib}^{\mathcal{H}}(S) := \left(S, X^{\mathcal{H}}(S), (p^{\mathcal{H}}(s))_{s \in S}\right),$$

where

$$X^{\mathcal{H}}(S) := \coprod_{f \in \Gamma(S)} \left[\prod_{s \in S} G_{f(s)}(\mathcal{H})\right]$$

and, for every $s \in S$,

$$p^{\mathcal{H}}(s) := \coprod_{f \in \Gamma(S)} \left[p_{f(s)}(\mathcal{H}) \times \prod_{s' \in S \smallsetminus \{s\}} \mathrm{id}_{G_{f(s')}(\mathcal{H})}\right].$$

Let $\gamma : S \to T$ be a morphism in Γ. We define a morphism $\mathrm{Fib}^{\mathcal{H}}(\gamma) : \mathrm{Fib}^{\mathcal{H}}(T) \to \mathrm{Fib}^{\mathcal{H}}(S)$ of Γ-Fib_F for which $\mathcal{O}_\Gamma^F(\mathrm{Fib}^{\mathcal{H}}(\gamma)) = \gamma$, in the following way: for every $f \in \Gamma(S)$, we consider the map

$$\begin{cases} \prod_{t \in \gamma(s)} G_{f(t)}(\mathcal{H}) & \longrightarrow G_{f \circ \gamma(s)}(\mathcal{H}) \\ (E_t)_{t \in \gamma(s)} & \longmapsto \overset{\perp}{\underset{t \in \gamma(s)}{\oplus}} E_t, \end{cases}$$

and, for every $s \in S$ and $f \in \Gamma(S)$, we have a commutative square:

$$\begin{array}{ccc} \left[\underset{t \in \gamma(s)}{\oplus} E_{f(t)}(\mathcal{H})\right] \times \prod_{t \in T \smallsetminus \gamma(s)} G_{f(t)}(\mathcal{H}) & \longrightarrow & E_{f \circ \gamma(s)}(\mathcal{H}) \times \prod_{s' \in S \smallsetminus \{s\}} G_{f(s')}(\mathcal{H}) \\ \downarrow & & \downarrow \\ \prod_{t \in T} G_{f(t)}(\mathcal{H}) & \longrightarrow & \prod_{s_1 \in S} G_{f \circ \gamma(s_1)}(\mathcal{H}), \end{array}$$

where the upper morphism is given by the previous map and the following one:

$$\begin{cases} \underset{t \in \gamma(s)}{\oplus} E_{f(t)}(\mathcal{H}) & \longrightarrow E_{f \circ \gamma(s)}(\mathcal{H}) \\ (x_t)_{t \in \gamma(s)} & \longmapsto \sum_{t \in \gamma(s)} x_t. \end{cases}$$

The above squares turn out to define strong morphisms of Hilbert bundles, and we deduce that $\mathrm{Fib}^{\mathcal{H}}(\gamma)$ is a morphism in Γ-Fib_F whose image by $\mathcal{O}_\Gamma^F$ is γ. Checking that this is compatible with the composition of morphisms is an easy task, and we conclude that we have defined a functor

$$\mathrm{Fib}^{\mathcal{H}} : \Gamma \longrightarrow \Gamma\text{-}\mathrm{Fib}_F$$

such that $\mathcal{O}_\Gamma^F \circ \mathrm{Fib}^{\mathcal{H}} = \mathrm{id}_\Gamma$.

Remark 8. The underlying idea is that we want to map any finite family of finite dimensional subspaces of $\mathcal{H}^{(\infty)}$ to its orthogonal direct sum in $\mathcal{H}^{(\infty)}$, if possible. The problem is solved by taking "labeled" subspaces, i.e. pairs (A, x) consisting of a finite subset A of $\mathbb{N}$ and an $\#A$-dimensional subspace x of $\mathcal{H}^A$ (the condition on the dimension of x is not necessary in this paper but will prove crucial in [15]). We then only accept families $((A_i, x_i))_{i \in I}$ consisting of labeled subspaces such that the A_i's are pairwise disjoint. With this condition, the x_i's are automatically pairwise orthogonal, and their orthogonal direct sum may be seen as a subspace of $\mathcal{H}^{\underset{i \in I}{\cup} A_i}$ so that the pair $\left(\underset{i \in I}{\cup} A_i, \underset{i \in I}{\overset{\perp}{\oplus}} x_i \right)$ is a labeled subspace.

Proposition 3.9. *Let $\mathcal{H}$ be an inner product space with ground field F. Assume that $\mathcal{H}$ is finite-dimensional or isomorphic to $F^{(\infty)}$. Then $\mathrm{Fib}^{\mathcal{H}}$ is a Hilbert Γ-bundle.*

Proof. Condition (i) has already been checked. Moreover, $\mathrm{Fib}^{\mathcal{H}}(\mathbf{0}) = (\mathbf{0}, *, \emptyset)$, by construction. It remains to prove that condition (iii) is satisfied.

We choose n elements $f_1, f_2, ..., f_n$ in $\Gamma(\{1\})$. Every f_i then corresponds to a finite subset of $\mathbb{N}$. If $f_i(1) = \emptyset$, we set $\max(f_i(1)) = -1$. We then define

$$
\coprod_{1 \le i \le n} f_i : \begin{cases} \{1, \dots, n\} & \longrightarrow \mathcal{P}(\mathbb{N}) \\ 1 & \longmapsto f_1(1) \\ i > 1 & \longmapsto f_i(1) + i - 1 + \sum_{1 \le j \le i-1} \max(f_j(1)). \end{cases}
$$

For every $k \in \{1, \dots, n\}$, there is an increasing bijection from $f_k(1)$ to $\left(\coprod_{1 \le i \le n} f_i \right)(k)$, and these bijections give rise to a continuous map:

$$
\prod_{1 \le i \le n} G_{f_i(1)}(\mathcal{H}) \longrightarrow \prod_{1 \le i \le n} G_{\left(\coprod_{1 \le j \le n} f_j \right)(i)}(\mathcal{H}),
$$

and, for every $i \in \{1, \dots, n\}$, a strong morphism of Hilbert bundles:

$$
\begin{array}{ccc}
E_{f_i(1)}(\mathcal{H}) & \longrightarrow & E_{\left(\coprod_{1 \le j \le n} f_j \right)(i)}(\mathcal{H}) \\
\downarrow & & \downarrow \\
G_{f_i(1)}(\mathcal{H}) & \longrightarrow & G_{\left(\coprod_{1 \le j \le n} f_j \right)(i)}(\mathcal{H}).
\end{array}
$$

We have just constructed a morphism $n.\mathrm{Fib}^{\mathcal{H}}(1) \longrightarrow \mathrm{Fib}^{\mathcal{H}}(n)$ of Γ-Fib_F, whose image by $\mathcal{O}_\Gamma^F$ is $\mathrm{id}_{\mathbf{n}}$. Hence condition (iii) is satisfied. $\square$

We now tackle the special case where $\mathcal{H} = F^m$ for $1 \le m \le \infty$, and try to relate it to the constructions of Section 2 (when $m = \infty$, F^m should be

understood as $F^{(\infty)}$). We will build two morphisms $\mathrm{Fib}^{F^m}(\mathbf{1}) \leftrightarrows \gamma^{(m)}(F)$ the image of which is $\mathrm{id}_{\mathbf{1}}$ by $\mathcal{O}^F_\Gamma$ (cf. Section 2.4.2 for the definition of $\gamma^{(m)}(F)$).

For $n \in \mathbb{N}$, we can identify $\gamma_n^{mn}(F)$ with $p_{\{0,\ldots,n-1\}}(F^m)$. We then obtain a strong morphism of Hilbert bundles:

$$
\begin{array}{ccc}
\coprod\limits_{n\in\mathbb{N}} E_n(F^{mn}) & \longrightarrow & \coprod\limits_{f\in\Gamma(\mathbf{1})} E_{f(1)}(F^m) \\[2mm]
{\scriptstyle\gamma^{(m)}(F)}\Big\downarrow & & \Big\downarrow{\scriptstyle\coprod\limits_{f\in\Gamma(\mathbf{1})} p_{f(1)}(F^m)} \\[2mm]
\coprod\limits_{n\in\mathbb{N}} G_n(F^{mn}) & \longrightarrow & \coprod\limits_{f\in\Gamma(\mathbf{1})} G_{f(1)}(F^m),
\end{array}
$$

defining our first morphism $\gamma^{(m)}(F) \longrightarrow \mathrm{Fib}^{F^m}(\mathbf{1})$.

For any element $f \in \Gamma(\mathbf{1})$, we choose a bijection from $f(1)$ to $\{1,\ldots,n\}$ (for some $n \in \mathbb{N}$). This induces a strong morphism of Hilbert bundles:

$$
\begin{array}{ccc}
E_{f(1)}(F^m) & \longrightarrow & E_n(F^{mn}) \\[2mm]
\Big\downarrow & & \Big\downarrow \\[2mm]
G_{f(1)}(F^m) & \longrightarrow & G_n(F^{mn}).
\end{array}
$$

Collecting those strong morphisms yields the second morphism $\mathrm{Fib}^{F^m}(\mathbf{1}) \longrightarrow \gamma^{(m)}(F)$. We may now set:

$$\underline{\mathrm{Vec}}^{F,m}_G := \underline{\mathrm{Vec}}^{\mathrm{Fib}^{F^m}}_G \quad ; \quad \mathrm{Vec}^{F,m}_G := \mathrm{Vec}^{\mathrm{Fib}^{F^m}(\mathbf{1})}_G \quad ; \quad E\,\mathrm{Vec}^{F,m}_G := E\,\mathrm{Vec}^{\mathrm{Fib}^{F^m}(\mathbf{1})}_G;$$

$$i\underline{\mathrm{Vec}}^{F,m}_G := i\underline{\mathrm{Vec}}^{\mathrm{Fib}^{F^m}}_G \quad ; \quad i\,\mathrm{Vec}^{F,m}_G := i\,\mathrm{Vec}^{\mathrm{Fib}^{F^m}(\mathbf{1})}_G \quad ; \quad Ei\,\mathrm{Vec}^{F,m}_G := Ei\,\mathrm{Vec}^{\mathrm{Fib}^{F^m}(\mathbf{1})}_G,$$

and

$$s\underline{\mathrm{Vec}}^{F,m}_G := s\underline{\mathrm{Vec}}^{\mathrm{Fib}^{F^m}}_G \quad ; \quad s\,\mathrm{Vec}^{F,m}_G := s\,\mathrm{Vec}^{\mathrm{Fib}^{F^m}(\mathbf{1})}_G \quad ; \quad Es\,\mathrm{Vec}^{F,m}_G := Es\,\mathrm{Vec}^{\mathrm{Fib}^{F^m}(\mathbf{1})}_G .$$

The Γ-space structure of $\underline{\mathrm{Vec}}^{F,m}_G$ induces a structure of equivariant H-space on $\mathrm{Vec}^{F,m}_G = \underline{\mathrm{Vec}}^{F,m}_G(\mathbf{1})$.

Proposition 3.10. *Let G be a second-countable Lie group and $m \in \mathbb{N}^* \cup \{\infty\}$ such that $m \geq \dim G$. Then $E\,\mathrm{Vec}^{F,m}_G \to \mathrm{Vec}^{F,m}_G$ is universal for finite-dimensional G-vector bundles over G-CW-complexes, and, for every G-CW-complex X, the induced bijection*

$$\Phi : [X, \mathrm{Vec}^{F,m}_G]_G \xrightarrow{\cong} \mathbb{Vect}^F_G(X)$$

is a homomorphism of abelian monoids.

Remark 9. Similar results hold for $i\,\mathrm{Vec}^{F,m}_G$ (respectively for $s\,\mathrm{Vec}^{F,m}_G$) by replacing the notion of finite-dimensional G-vector bundle by the notion of G-Hilbert bundle (resp. by the notion of G-simi-Hilbert bundle).

Before proving this, we need two lemmas (the proofs are straightforward hence omitted):

Lemma 3.11. *Let*

$$
\begin{array}{ccc}
E & \longrightarrow & E' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
X & \longrightarrow & X'
\end{array}
$$

be a strong morphism of Hilbert bundles. Then the induced square

$$
\begin{array}{ccc}
E\operatorname{Vec}_G^p & \longrightarrow & E\operatorname{Vec}_G^q \\
\downarrow & & \downarrow \\
\operatorname{Vec}_G^p & \longrightarrow & \operatorname{Vec}_G^q
\end{array}
$$

is a strong morphism of finite-dimensional G-vector bundles.

Lemma 3.12. *Let $\varphi = (S, X, (p_s)_{s\in S}) \xrightarrow{f} \psi = (T, Y, (q_t)_{t\in T})$ be a morphism in $\Gamma - Fib_F$ such that $\mathcal{O}_\Gamma^F(f)(T) = S$. Then f induces a commutative square*

$$
\begin{array}{ccc}
|\operatorname{Func}(\mathcal{E}G, \varphi\text{-Bdl})| & \longrightarrow & |\operatorname{Func}(\mathcal{E}G, \psi\text{-Bdl})| \\
\downarrow & & \downarrow \\
|\operatorname{Func}(\mathcal{E}G, \varphi\text{-mod})| & \longrightarrow & |\operatorname{Func}(\mathcal{E}G, \psi\text{-mod})|
\end{array}
$$

which defines a strong morphism of finite-dimensional pseudo-G-vector bundles.

Proof of Proposition . From the previous constructions and Proposition 3.2, we deduce that the G-maps $\operatorname{Vec}_G^{F,m} \to \operatorname{Vec}_G^{\gamma^{(m)}(F)}$ and $\operatorname{Vec}_G^{\gamma^{(m)}(F)} \to \operatorname{Vec}_G^{F,m}$ (induced by the above functors) are equivariant homotopy equivalences that are inverse one to the other up to an equivariant homotopy.

Let X be a G-CW-complex. We deduce that the map $[X, \operatorname{Vec}_G^{\gamma^{(m)}(F)}]_G \xrightarrow{\cong} [X, \operatorname{Vec}_G^{F,m}]_G$ induced by $\operatorname{Vec}_G^{F,m} \to \operatorname{Vec}_G^{\gamma^{(m)}(F)}$ is a bijection. However, since Lemma 3.11 shows the square

$$
\begin{array}{ccc}
E\operatorname{Vec}_G^{\gamma^{(m)}(F)} & \longrightarrow & E\operatorname{Vec}_G^{F,m} \\
\downarrow & & \downarrow \\
\operatorname{Vec}_G^{\gamma^{(m)}(F)} & \longrightarrow & \operatorname{Vec}_G^{F,m}
\end{array}
$$

is a strong morphism of finite-dimensional G-vector bundles, the composite of $[X, \operatorname{Vec}_G^{\gamma^{(m)}(F)}]_G \xrightarrow{\cong} [X, \operatorname{Vec}_G^{F,m}]_G$ with $\left[X, \operatorname{Vec}_G^{F,m}\right]_G \longrightarrow \mathbb{V}\mathrm{ect}_G^F(X)$ (induced by pulling back the G-vector bundle $E\operatorname{Vec}_G^{F,m} \to \operatorname{Vec}_G^{F,m}$) is the map $\left[X, \operatorname{Vec}_G^{\gamma^{(m)}(F)}\right]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^F(X)$ induced by pulling back the G-vector bundle $E\operatorname{Vec}_G^{\gamma^{(m)}(F)} \to \operatorname{Vec}_G^{\gamma^{(m)}(F)}$.

We know from Section 2.4.2 that this last map is a bijection, and we deduce that the G-vector bundle $E\operatorname{Vec}_G^{F,m} \to \operatorname{Vec}_G^{F,m}$ induces a bijection $\Phi : \left[X, \operatorname{Vec}_G^{F,m}\right]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^F(X)$. This proves the first part of the proposition.

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In order to prove the second part, we consider the commutative square:

$$E\operatorname{Vec}_G^{F,m}\underset{\mathrm{k}}{\times}E\operatorname{Vec}_G^{F,m}\longrightarrow\left|\operatorname{Func}(\mathcal{E}G,(\operatorname{Fib}^{F^m}(\mathbf{1})\text{-Bdl})^2)\right|$$
$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$
$$\operatorname{Vec}_G^{F,m}\underset{\mathrm{k}}{\times}\operatorname{Vec}_G^{F,m}\longrightarrow\left|\operatorname{Func}(\mathcal{E}G,(\operatorname{Fib}^{F^m}(\mathbf{1})\text{-mod})^2)\right|,$$

which defines a strong morphism of pseudo-G-vector bundles; and

$$\left|\operatorname{Func}(\mathcal{E}G,(\operatorname{Fib}^{F^m}(\mathbf{1})\text{-Bdl})^2)\right|\longrightarrow\left|\operatorname{Func}(\mathcal{E}G,(2.\operatorname{Fib}^{F^m}(\mathbf{1}))\text{-Bdl})\right|$$
$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$
$$\left|\operatorname{Func}(\mathcal{E}G,(\operatorname{Fib}^{F^m}(\mathbf{1})\text{-mod})^2)\right|\longrightarrow\left|\operatorname{Func}(\mathcal{E}G,(2.\operatorname{Fib}^{F^m}(\mathbf{1}))\text{-mod})\right|,$$

which also defines a strong morphism of pseudo-G-vector bundles.

Since Fib^{F^m} is a Hilbert Γ-bundle, we have a functor $(\operatorname{Fib}^{F^m}(\mathbf{1}))^2\longrightarrow\operatorname{Fib}^{F^m}(\mathbf{2})$ (by condition (iii)), and a functor $\operatorname{Fib}^{F^m}(\mathbf{2})\to\operatorname{Fib}^{F^m}(\mathbf{1})$ induced by the morphism $\{1\}\to\{1,2\}$ of Γ which sends 1 to $\{1,2\}$. The composition of these functors induces a commutative square

$$\left|\operatorname{Func}(\mathcal{E}G,(\operatorname{Fib}^{F^m}(\mathbf{1}))^2\text{-Bdl})\right|\longrightarrow\left|\operatorname{Func}(\mathcal{E}G,\operatorname{Fib}^{F^m}(\mathbf{1})\text{-Bdl})\right|$$
$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$
$$\left|\operatorname{Func}(\mathcal{E}G,(\operatorname{Fib}^{F^m}(\mathbf{1}))^2\text{-mod})\right|\longrightarrow\left|\operatorname{Func}(\mathcal{E}G,\operatorname{Fib}^{F^m}(\mathbf{1})\text{-mod})\right|,$$

and Lemma 3.12 shows that it defines a strong morphism of pseudo-G-vector bundles.

By composing these three squares, we finally obtain a square

$$E\operatorname{Vec}_G^{F,m}\underset{\mathrm{k}}{\times}E\operatorname{Vec}_G^{F,m}\longrightarrow E\operatorname{Vec}_G^{F,m}$$
$$\downarrow\qquad\qquad\qquad\qquad\downarrow$$
$$\operatorname{Vec}_G^{F,m}\underset{\mathrm{k}}{\times}\operatorname{Vec}_G^{F,m}\overset{\pi}{\longrightarrow}\operatorname{Vec}_G^{F,m},$$

which is a strong morphism of pseudo-G-vector bundles, and where, by construction, the map π arises from the equivariant H-space structure on $\operatorname{Vec}_G^{F,m}$ induced by the equivariant Γ-space structure on $\underline{\operatorname{Vec}}_G^{F,m}$.

Let $p:E\to X$ and $q:E'\to X$ be two G-vector bundles over a G-CW-complex X, and $f:X\to\operatorname{Vec}_G^{F,m}$ (resp. $g:X\to\operatorname{Vec}_G^{F,m}$) be a map corresponding to $[p]$ (resp. $[q]$) by the isomorphism $\Phi:[X,\operatorname{Vec}_G^{F,m}]_G\overset{\cong}{\longrightarrow}\mathbb{V}\mathrm{ect}_G^F(X)$. Then the

square

$$E \oplus E' \longrightarrow E\operatorname{Vec}_G^{F,m} \underset{\mathrm{k}}{\times} E\operatorname{Vec}_G^{F,m}$$

$$p \oplus q \downarrow \qquad\qquad \downarrow$$

$$X \xrightarrow{\ (f,g)\ } \operatorname{Vec}_G^{F,m} \underset{\mathrm{k}}{\times} \operatorname{Vec}_G^{F,m}$$

defines a strong morphism of pseudo-G-vector bundles. Right-composing it with the preceding square yields a new square

$$E \oplus E' \longrightarrow E\operatorname{Vec}_G^{F,m}$$

$$\downarrow \qquad\qquad \downarrow$$

$$X \xrightarrow{\ \pi\circ(f,g)\ } \operatorname{Vec}_G^{F,m}$$

which defines a strong morphism of finite-dimensional G-Hilbert bundles, and we deduce that $\Phi([\pi \circ (f,g)]) = [p \oplus q]$. We conclude that

$$\Phi([f] + [g]) = \Phi([\pi \circ (f,g)]) = [p \oplus q] = [p] + [q] = \Phi([f]) + \Phi([g]).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

3.4.4 Relating the three constructions

Here, we will assume that the Lie group G is second-countable, i.e. that $\pi_0(G)$ is countable.

The natural inclusions $U_n(F) \subset \operatorname{GU}_n(F) \subset \operatorname{GL}_n(F)$ induce a commutative diagram of natural transformations:

$$\text{-iBdl} \longrightarrow \text{-sBdl} \longrightarrow \text{-Bdl}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\text{-imod} \longrightarrow \text{-smod} \longrightarrow \text{-mod}.$$

For every $m \in \mathbb{N}^* \cup \{\infty\}$, we have the following diagram:

$$Ei\operatorname{Vec}_G^{F,m} \longrightarrow Es\operatorname{Vec}_G^{F,m} \longrightarrow E\operatorname{Vec}_G^{F,m}$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$i\operatorname{Vec}_G^{F,m} \longrightarrow s\operatorname{Vec}_G^{F,m} \longrightarrow \operatorname{Vec}_G^{F,m}.$$

The left-hand square is a strong morphism of G-simi-Hilbert bundles, whilst the right-hand square is a strong morphism of G-vector bundles. This yields natural transformations

$$[-, i\operatorname{Vec}_G^{F,m}]_G \to [-, s\operatorname{Vec}_G^{F,m}]_G \to [-, \operatorname{Vec}_G^{F,m}]_G$$

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on the category of G-spaces. We therefore obtain a commutative diagram

$$
\begin{array}{ccccc}
[-, i\operatorname{Vec}_G^{F,m}]_G & \longrightarrow & [-, s\operatorname{Vec}_G^{F,m}]_G & \longrightarrow & [-, \operatorname{Vec}_G^{F,m}]_G \\
\Big\downarrow{i\Phi} & & \Big\downarrow{s\Phi} & & \Big\downarrow{\Phi} \\
i\mathbb{V}\mathrm{ect}_G^F(-) & \longrightarrow & s\mathbb{V}\mathrm{ect}_G^F(-) & \longrightarrow & \mathbb{V}\mathrm{ect}_G^F(-)
\end{array}
\tag{1}
$$

By Proposition 3.4.3, Φ, $i\Phi$ and $s\Phi$ are isomorphisms on the category of G-CW-complexes for any $m \geq \dim G$, We want to prove that this diagram consists solely of isomorphisms on the category of *proper* G-CW-complexes whenever $m \geq \dim(G)$.

Lemma 3.13. *Let X be a finite proper G-CW-complex, and G be a second countable Lie group. Then*

$$
i\mathbb{V}\mathrm{ect}_G^F(X) \xrightarrow{\cong} s\mathbb{V}\mathrm{ect}_G^F(X) \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^F(X).
$$

Proof. Let X be a finite proper G-CW-complex. It suffices to prove the following statements.

  (i) Any G-vector bundle over X can be equipped with a compatible structure of G-Hilbert bundle.

  (ii) Any G-simi-Hilbert bundle can be equipped with a compatible structure of G-Hilbert bundle.

  (iii) If two G-Hilbert bundles $E \to X$ and $E' \to X$ are isomorphic as G-vector bundles, then they are isomorphic as G-Hilbert bundles.

Statement (i) was proven by Phillips (cf. example 3.4 of [12]) in the case $F = \mathbb{C}$, since X, being a finite proper G-CW-complex, is locally compact. The same technique also yields the case $F = \mathbb{R}$.

Statement (ii) may be derived from statement (i). Indeed, given an n-dimensional G-simi-Hilbert bundle $E \to X$, we want to find an equivariant section of the bundle $\bar{E} \to X$, where $\bar{E}_x$ is the linear family of inner products on E_x given by the G-simi-Hilbert space structure, for every $x \in X$. This is a $(G, \mathbb{R}_+^*)$-principal bundle. If $E' \to X$ is a $(G, \mathbb{R}_+^*)$-principal bundle, then putting a G-Hilbert bundle structure on the associated G-vector bundle is the same as finding a section of $E' \to X$. Hence, (ii) follows from (i) applied to the real case for 1-dimensional G-vector bundles.

Finally, let $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi'} X$ be two G-Hilbert bundles, and $f : E \to E'$ an isomorphism of G-vector bundles. Then $f(f^*f)^{-\frac{1}{2}} : E \to E'$ is an isomorphism of G-Hilbert bundles. This proves statement (iii). $\qquad\square$

Proposition 3.14. *Let G be a second-countable Lie group and let $m \geq \dim(G)$. Then the maps*
$$
i\operatorname{Vec}_G^{F,m} \to s\operatorname{Vec}_G^{F,m} \to \operatorname{Vec}_G^{F,m}
$$
are G-weak equivalences.

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Corollary 3.15. *Let G be a second-countable Lie group and $m \geq \dim(G)$. Then diagram (1) consists entirely of isomorphisms (on the category of proper G-CW-complexes).*

Proof of Corollary 3.15: Proposition 3.14 shows that, for every proper G-CW-complex X, there are two isomorphisms:

$$[X, i\operatorname{Vec}_G^{F,m}]_G \xrightarrow{\cong} [X, s\operatorname{Vec}_G^{F,m}]_G \xrightarrow{\cong} [X, \operatorname{Vec}_G^{F,m}]_G.$$

This is equivalent to the condition that the two upper horizontal morphisms in diagram (1) be isomorphisms on the category of proper G-CW-complexes. Therefore, it is also true for the lower horizontal morphisms. $\square$

Proof of Proposition 3.14: By Lemma 3.13, the lower horizontal morphisms in (1) are isomorphisms on the category of finite proper G-CW-complexes. We deduce that the upper horizontal morphisms in (1) are isomorphisms on the category of finite proper G-CW-complexes. Therefore, for every $n \in \mathbb{N}$ and every compact subgroup H of G,

$$[G/H \times S^n, i\operatorname{Vec}_G^{F,m}]_G \xrightarrow{\cong} [G/H \times S^n, s\operatorname{Vec}_G^{F,m}]_G \xrightarrow{\cong} [G/H \times S^n, \operatorname{Vec}_G^{F,m}]_G,$$

hence
$$\pi_n\big((i\operatorname{Vec}_G^{F,m})^H\big) \xrightarrow{\cong} \pi_n\big((s\operatorname{Vec}_G^{F,m})^H\big) \xrightarrow{\cong} \pi_n\big((\operatorname{Vec}_G^{F,m})^H\big)$$

for every $n \in \mathbb{N}$ and every compact subgroup H of G. $\square$

3.4.5 How the constructions depend on m

If m and m' are distinct elements of $\mathbb{N}^* \cup \{\infty\}$, there are linear injections of F^m into $F^{m'}$. We investigate here what they induce on the preceding constructions.

Let φ and φ' be two Hilbert Γ-bundles. Any natural transformation $\eta : \varphi \to \varphi'$ induces natural transformations $\eta^* : \varphi\text{-mod} \to \varphi'\text{-mod}$, $i\eta^* : \varphi\text{-imod} \to \varphi'\text{-imod}$ and $s\eta^* : \varphi\text{-smod} \to \varphi'\text{-smod}$, which respectively induce homomorphisms of equivariant Γ-spaces $\eta^* : \underline{\operatorname{Vec}}_G^\varphi \to \underline{\operatorname{Vec}}_G^{\varphi'}$, $i\eta^* : i\underline{\operatorname{Vec}}_G^\varphi \to i\underline{\operatorname{Vec}}_G^{\varphi'}$, and $s\eta^* : s\underline{\operatorname{Vec}}_G^\varphi \to s\underline{\operatorname{Vec}}_G^{\varphi'}$.

Lemma 3.16. *Let φ be a Γ-Hilbert bundle and $\eta : \varphi \to \varphi$ be a natural transformation such that $\mathcal{O}_\Gamma^F(\eta) = \operatorname{id}_\Gamma$. Then there exists a natural transformation*

$$\underline{\operatorname{Vec}}_G^{F,\varphi} \times I \to \underline{\operatorname{Vec}}_G^{F,\varphi}$$

which is an equivariant homotopy from η^* to the identity map.
There are similar results for $i\eta^*$ and $s\eta^*$.

Proof. For every object S of Γ, the map $\begin{cases} \operatorname{Ob}(\varphi(S)\text{-mod}) = X_S & \longrightarrow \operatorname{Hom}(\varphi(S)\text{-mod}) \\ x & \longmapsto ((\eta_s)_{|x})_{s \in S} \end{cases}$
provides a natural transformation from the identity of $\varphi(S)$-mod to the functor $\eta^* : \varphi(S)\text{-mod} \to \varphi(S)\text{-mod}$. $\square$

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Let $\mathcal{H}$ and $\mathcal{H}'$ be two inner product spaces which are either finite-dimensional or isomorphic to $F^{(\infty)}$, together with an isometry $\alpha : \mathcal{H} \hookrightarrow \mathcal{H}'$ and an injective map $\beta : \mathbb{N} \hookrightarrow \mathbb{N}$. Then, for every finite subset A of $\mathbb{N}$, α and β induce a linear injection $\begin{cases} \mathcal{H}^A & \longrightarrow (\mathcal{H}')^{\beta(A)} \\ (x_a)_{a \in A} & \longmapsto \big(\alpha(x_{\beta^{-1}(b)})\big)_{b \in \beta(A)} \end{cases}$ and then a strong morphism of Hilbert bundles

$$
\begin{array}{ccc}
E_A(\mathcal{H}) & \longrightarrow & E_{\beta(A)}(\mathcal{H}') \\
\downarrow & & \downarrow \\
B_A(\mathcal{H}) & \longrightarrow & B_{\beta(A)}(\mathcal{H}').
\end{array}
$$

From those squares may be derived a natural transformation:

$$
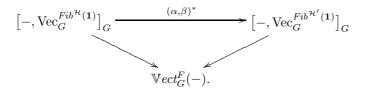(\alpha, \beta)^* : \mathrm{Fib}^{\mathcal{H}} \longrightarrow \mathrm{Fib}^{\mathcal{H}'}.
$$

This yields natural transformations

$$
(\alpha, \beta)^* : \underline{\mathrm{Vec}}_G^{\mathrm{Fib}^{\mathcal{H}}} \longrightarrow \underline{\mathrm{Vec}}_G^{\mathrm{Fib}^{\mathcal{H}'}} \quad ; \quad B(\alpha, \beta)^* : B\,\mathrm{Vec}_G^{\mathrm{Fib}^{\mathcal{H}}} \longrightarrow B\,\mathrm{Vec}_G^{\mathrm{Fib}^{\mathcal{H}'}},
$$

$$
i(\alpha, \beta)^* : i\underline{\mathrm{Vec}}_G^{\mathrm{Fib}^{\mathcal{H}}} \longrightarrow i\underline{\mathrm{Vec}}_G^{\mathrm{Fib}^{\mathcal{H}'}} \quad ; \quad Bi(\alpha, \beta)^* : Bi\,\mathrm{Vec}_G^{\mathrm{Fib}^{\mathcal{H}}} \longrightarrow Bi\,\mathrm{Vec}_G^{\mathrm{Fib}^{\mathcal{H}'}},
$$

and

$$
s(\alpha, \beta)^* : s\underline{\mathrm{Vec}}_G^{\mathrm{Fib}^{\mathcal{H}}} \longrightarrow s\underline{\mathrm{Vec}}_G^{\mathrm{Fib}^{\mathcal{H}'}} \quad ; \quad Bs(\alpha, \beta)^* : Bs\,\mathrm{Vec}_G^{\mathrm{Fib}^{\mathcal{H}}} \longrightarrow Bs\,\mathrm{Vec}_G^{\mathrm{Fib}^{\mathcal{H}'}} .
$$

Proposition 3.17. *Let $\alpha : \mathcal{H} \hookrightarrow \mathcal{H}'$ be an isometry between two inner product spaces which are either finite-dimensional or isomorphic to $F^{(\infty)}$, and let $\beta : \mathbb{N} \hookrightarrow \mathbb{N}$ be an injection.*

(a) *If $\mathcal{H} = \mathcal{H}'$, then $B(\alpha, \beta)^*$ is G-homotopic to the identity map, and so are $Bi(\alpha, \beta)^*$ and $Bs(\alpha, \beta)^*$.*

(b) *If $\alpha' : \mathcal{H} \hookrightarrow \mathcal{H}'$ is another isometry and $\beta' : \mathbb{N} \hookrightarrow \mathbb{N}$ is another injection, then $B(\alpha, \beta)^*$ and $B(\alpha', \beta')^*$ are G-homotopic (and the same result holds for $Bi(\alpha, \beta)^*$ and $Bi(\alpha', \beta')$ on the one hand, and $Bs(\alpha, \beta)^*$ and $Bs(\alpha', \beta')^*$ on the other hand).*

(c) *We have a commutative diagram:*

$$
\begin{array}{ccc}
\big[-, \mathrm{Vec}_G^{Fib^{\mathcal{H}}(\mathbf{1})}\big]_G & \xrightarrow{\;\;(\alpha,\beta)^*\;\;} & \big[-, \mathrm{Vec}_G^{Fib^{\mathcal{H}'}(\mathbf{1})}\big]_G \\
& \searrow \qquad \swarrow & \\
& \mathbb{V}ect_G^F(-). &
\end{array}
$$

Proof. We only prove the result in the case of $B(\alpha, \beta)^*$, since the other cases may be treated in a strictly identical manner.

(a) We apply Lemma 3.16 to the natural transformation $(\alpha, \beta)^*$. The homotopy from $B(\alpha, \beta)^*$ to the identity map of $B\operatorname{Vec}_G^\varphi$ is obtained by taking the geometric realization of the natural transformation $\underline{\operatorname{Vec}}_G^\varphi \times I \to \underline{\operatorname{Vec}}_G^\varphi$.

(b) Assume, for example, that $\#(\mathbb{N} \smallsetminus \beta'(\mathbb{N})) \geq \#(\mathbb{N} \smallsetminus \beta(\mathbb{N}))$. We then choose an injection $\beta'' : \mathbb{N} \hookrightarrow \mathbb{N}$ such that $\beta' = \beta'' \circ \beta$. Thus $B(\alpha, \beta') = B(\operatorname{id}_{\mathcal{H}'}, \beta'') \circ B(\alpha, \beta)$, and we deduce from (a) that $B(\alpha, \beta')$ is G-homotopic to $B(\alpha, \beta)$. It thus suffices to prove the result when $\beta = \beta'$.

Assume now that $\beta = \beta'$. If $\mathcal{H}$ is isomorphic to $F^{(\infty)}$, then $\mathcal{H}'$ is also isomorphic to $F^{(\infty)}$, and the result follows from (a). Assume finally that $\mathcal{H}$ is finite-dimensional. Then there exists an isometry $\alpha'' : \mathcal{H}' \stackrel{\cong}{\to} \mathcal{H}'$ such that $\alpha' = \alpha'' \circ \alpha$. Thus $B(\alpha', \beta) = B(\alpha'', \operatorname{id}_{\mathbb{N}}) \circ B(\alpha, \beta)$, and we deduce from (a) that $B(\alpha', \beta)$ is G-homotopic to $B(\alpha, \beta)$.

(c) Lemma 3.11 shows indeed that $(\alpha, \beta)^*$ induces a strong morphism of G-vector bundles:

$$
\begin{array}{ccc}
E\operatorname{Vec}_G^{\operatorname{Fib}^{\mathcal{H}}(\mathbf{1})} & \longrightarrow & E\operatorname{Vec}_G^{\operatorname{Fib}^{\mathcal{H}'}(\mathbf{1})} \\
\downarrow & & \downarrow \\
\operatorname{Vec}_G^{\operatorname{Fib}^{\mathcal{H}}(\mathbf{1})} & \longrightarrow & \operatorname{Vec}_G^{\operatorname{Fib}^{\mathcal{H}'}(\mathbf{1})}.
\end{array}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 10. If $m \geq \dim(G)$, then $B(\alpha, \beta)^*$, $Bi(\alpha, \beta)^*$ and $Bs(\alpha, \beta)^*$ are G-weak equivalences. This will only be used in the case $\mathcal{H} = \mathcal{H}'$, and it is then an immediate consequence of Proposition 3.17 (notice that the assumption $\dim(G) \leq m$ is useless in this case), so we leave the general case as an easy exercise.

4 Equivariant K-theory

In this chapter, $F = \mathbb{R}$ or $\mathbb{C}$. Recall that any $\Gamma - G$-space $\underline{A}$ has an underlying structure of simplicial G-space, and thus has a thick geometric realization BA, called the classifying space of $\underline{A}$, which inherits a structure of G-space.

For any Lie group G, and any $m \in \mathbb{N}^* \cup \{\infty\}$, we set:

$$
KF_G^{[m]} := \Omega B\operatorname{Vec}_G^{F,m} \quad ; \quad iKF_G^{[m]} := \Omega Bi\operatorname{Vec}_G^{F,m} \quad ; \quad sKF_G^{[m]} := \Omega Bs\operatorname{Vec}_G^{F,m} .
$$

In Section 4.1, we define the equivariant K-theory $KF_G^*(-)$ as the good equivariant cohomology theory in negative degrees classified by the equivariant H-space $KF_G^{[\infty]}$. In Section 4.2, we construct a natural transformation $\gamma : \mathbb{K}F_G^*(-) \to KF_G^*(-)$, and then prove in Section 4.3 that γ_X is an isomorphism whenever $X = (G/H) \times Y$, where H is a compact subgroup of G and Y is a finite CW-complex on which G acts trivially. The proof is based upon Segal's theorem on Γ-spaces. Alternative classifying spaces are discussed in Section 4.4. Finally, Sections 4.5 and 4.6 are devoted to the construction of product structures on $KF_G^*(-)$, on Bott periodicity and the extension to positive degrees.

4.1 Equivariant K-theory in negative degrees

Definition 4.1. Let G be a Lie group, $F = \mathbb{R}$ or $\mathbb{C}$, (X, A) a G-CW-pair and $n \in \mathbb{N}$. We set:

$$KF_G^{-n}(X, A) := \left[\Sigma^n (X/A), KF_G^{[\infty]} \right]_G^\bullet,$$

and

$$KF_G^{-n}(X) := KF_G^{-n}(X \cup \{*\}, \{*\}).$$

In particular, for every G-CW-complex X,

$$KF_G(X) := KF_G^0(X) = \left[X, KF_G^{[\infty]} \right]_G.$$

Proposition 4.2. $KF_G^*(-)$ *is a good equivariant cohomology theory in negative degrees on the category of G-CW-pairs.*

Proof. Both the excision property and the invariance under equivariant homotopy are obvious. The long exact sequence of a G-CW-pair is derived from the equivariant Puppe sequence associated to a G-cofibration, since the inclusion $A \subset X$ is a G-cofibration for every G-CW-pair (X, A) (for a reference on the non-equivariant Puppe sequence, cf. p.398 of [4]). $\qquad\square$

Remark 11. It follows from the properties of negatively-graded good equivariant cohomology theories that, for every G-CW-pair (X, A) and every $n \in \mathbb{N}$, we have a natural isomorphism

$$KF_G^{-n}(X) \cong \mathrm{Ker} \left[KF_G(S^n \times X) \longrightarrow KF_G(X) \right],$$

induced by the projection $S^n \times X \to (S^n \times X)/(\{*\} \times X)$, and a natural isomorphism

$$KF_G^{-n}(X, A) \cong \mathrm{Ker} \left[KF_G^{-n}(X \cup_A X) \longrightarrow KF_G^{-n}(X) \right],$$

induced by the projection $X \cup_A X \to X/A$.

4.2 The natural transformation $\gamma : \mathbb{K}F_G^*(-) \to KF_G^*(-)$

Let X be a G-CW-complex. Then the canonical map $i : \mathrm{Vec}_G^{F,\infty} = \underline{\mathrm{Vec}}_G^{F,\infty}(\mathbf{1}) \to KF_G^{[\infty]}$ (which is a homomorphism of equivariant H-spaces) induces a natural homomorphism $[X, \mathrm{Vec}_G^{F,\infty}]_G \xrightarrow{i_*} [X, KF_G^{[\infty]}]_G$ of abelian monoids, which, precomposed with the inverse of the natural isomorphism $[X, \mathrm{Vec}_G^{F,\infty}]_G \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_G^F(X)$, yields a natural homomorphism of abelian monoids

$$\mathbb{V}\mathrm{ect}_G^F(X) \longrightarrow [X, KF_G^{[\infty]}]_G.$$

Since $[X, KF_G^{[\infty]}]_G$ is an abelian group for any G-space X, we deduce from the universal property of the Grothendieck construction that this morphism induces a natural homomorphism of abelian groups

$$\gamma_X : \mathbb{K}F_G(X) \longrightarrow KF_G(X).$$

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This clearly defines a natural transformation $\gamma : \mathbb{K}F_G(-) \longrightarrow KF_G(-)$ on the category of G-CW-complexes. For every G-CW-complex X and every $n \in \mathbb{N}$, the inclusion of X into $S^n \times X$ induces a commutative square:

$$
\begin{array}{ccc}
\mathbb{K}F_G(S^n \times X) & \longrightarrow & \mathbb{K}F_G(X) \\
\gamma_{S^n \times X} \downarrow & & \gamma_X \downarrow \\
KF_G(S^n \times X) & \longrightarrow & KF_G(X).
\end{array}
$$

However, we know from Remark 11 that $KF_G^{-n}(X) \cong \ker\left[KF_G(S^n \times X) \longrightarrow KF_G(X)\right]$ whilst, by definition, $\mathbb{K}F_G^{-n}(X) = \mathrm{Ker}\left[\mathbb{K}F_G(S^n \times X) \longrightarrow \mathbb{K}F_G(X)\right]$ (cf. [8] definition 3.1). The preceding square thus induces a functorial homomorphism:

$$
\gamma_X^{-n} : \mathbb{K}F_G^{-n}(X) \longrightarrow KF_G^{-n}(X).
$$

Finally, given a G-CW-pair (X, A) and an integer $n \in \mathbb{N}$, the inclusion of X into $X \cup_A X$ induces a commutative square:

$$
\begin{array}{ccc}
\mathbb{K}F_G^{-n}(X \cup_A X) & \longrightarrow & \mathbb{K}F_G^{-n}(X) \\
\gamma_{X \cup_A X}^{-n} \downarrow & & \gamma_X^{-n} \downarrow \\
KF_G^{-n}(X \cup_A X) & \longrightarrow & KF_G^{-n}(X).
\end{array}
$$

Again, Remark 11 shows $KF_G^{-n}(X, A) \cong \mathrm{Ker}\left[KF_G^{-n}(X \cup_A X) \longrightarrow KF_G^{-n}(X)\right]$ whilst, by definition, $\mathbb{K}F_G^{-n}(X, A) = \mathrm{Ker}\left[\mathbb{K}F_G^{-n}(X \cup_A X) \longrightarrow \mathbb{K}F_G^{-n}(X)\right]$. The previous square thus induces a functorial homomorphism:

$$
\gamma_{X,A}^{-n} : \mathbb{K}F_G^{-n}(X, A) \longrightarrow KF_G^{-n}(X, A).
$$

This completes the definition of γ as a natural transformation between negatively-graded equivariant cohomology theories on the category of G-CW-complexes.

4.3 The coefficients of $KF_G^*(-)$

The next results will show that $KF_G^*(-)$ deserves the label "equivariant K-theory".

Proposition 4.3. *Let H be a compact subgroup of G, and Y be a finite CW-complex on which G acts trivially. Then the map*

$$
\gamma_{(G/H) \times Y} : \mathbb{K}F_G((G/H) \times Y) \xrightarrow{\cong} \left[(G/H) \times Y, KF_G^{[\infty]}\right]_G
$$

is an isomorphism.

Corollary 4.4. *For every compact subgroup H of G, every $n \in \mathbb{N}$ and every finite CW-complex Y on which G acts trivially, the map*

$$
\gamma_{(G/H) \times Y}^{-n} : \mathbb{K}F_G^{-n}((G/H) \times Y) \xrightarrow{\cong} KF_G^{-n}((G/H) \times Y)
$$

is an isomorphism.

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Proof of Corollary 4.4: The case $n = 0$ is precisely the result of Proposition 4.3. For $n > 0$, we notice that if Y is a finite CW-complex on which G acts trivially, then so is $S^n \times Y$. We thus have a commutative square

$$
\begin{array}{ccc}
\mathbb{K}F_G(S^n \times ((G/H) \times Y)) & \longrightarrow & \mathbb{K}F_G((G/H) \times Y) \\
\cong \downarrow \gamma_{S^n \times ((G/H) \times Y)} & & \cong \downarrow \gamma_{(G/H) \times Y} \\
KF_G(S^n \times ((G/H) \times Y)) & \longrightarrow & KF_G((G/H) \times Y).
\end{array}
$$

We deduce that $\gamma^{-n}_{(G/H) \times Y}$ is an isomorphism. $\qquad\qquad\square$

In order to prove Proposition 4.3, we need to establish the following result:

Proposition 4.5. *Let H be a compact subgroup of a Lie group G. Then $\mathbb{K}F_G((G/H) \times -)$ is classified by an H-space as a functor from the category of finite CW-complexes to the category of abelian groups.*

Proof. Notice first that for any finite CW-complex Y on which G acts trivially, there is a natural isomorphism

$$
\begin{cases}
\mathbb{V}\mathrm{ect}^F_H(Y) & \xrightarrow{\cong} \mathbb{V}\mathrm{ect}^F_G((G/H) \times Y) \\
[E] & \longmapsto [G \times_H E].
\end{cases}
$$

This yields a functorial isomorphism

$$
\mathbb{K}F_H(-) \xrightarrow{\cong} \mathbb{K}F_G((G/H) \times -)
$$

on the category of finite CW-complexes on which G acts trivially. It thus all comes down to proving that $\mathbb{K}F_H$ is classified by an H-space over the category of finite CW-complexes on which H acts trivially. In the rest of the proof, we let J denote the set of isomorphism classes of irreducible finite-dimensional linear representation of H with ground field F. Since H is a compact group, we know that J is countable.
• **The case $F = \mathbb{C}$.** For every $j \in J$, we choose some V_j in the class j. For any finite CW-complex Y on which H acts trivially, we have a canonical H-vector bundle $\xi_{V_j}(Y) : V_j \times Y \to Y$ over Y. Next, we consider the two morphisms

$$
\begin{cases}
\mathbb{V}\mathrm{ect}_H(Y) & \longrightarrow \bigoplus_{j \in J} \mathbb{V}\mathrm{ect}(Y) \\
\xi & \longmapsto \underset{j \in J}{\oplus} \mathrm{Hom}(\xi_{V_j}(Y), \xi)
\end{cases}
\quad \text{and} \quad
\begin{cases}
\bigoplus_{j \in J} \mathbb{V}\mathrm{ect}(Y) & \longrightarrow \mathbb{V}\mathrm{ect}_H(Y) \\
(\xi_j)_{j \in J} & \longmapsto \underset{j \in J}{\oplus} (\xi_j \otimes \xi_{V_j}(Y)).
\end{cases}
$$

The theory of linear representations of compact groups shows that these homomorphisms are well-defined and inverse one to the other. They are also functorial, and so yield a natural isomorphism

$$
\mathbb{K}_H(Y) \xrightarrow{\cong} \underset{j \in J}{\oplus} \mathbb{K}(Y).
$$

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We choose an H-space B which classifies $\mathbb{K}(-)$ on the category of compact spaces, with a strict unit e (i.e. $\forall x \in B$, $x.e = e.x = x$), e.g. $B = \mathbb{Z} \times \underset{n \in \mathbb{N}}{\mathrm{colim}} B\mathrm{GL}_n(\mathbb{C})$.

If J is finite of order N, then B^N is the sought H-space.

Assume now that J is infinite. Then $\mathbb{K}F_H(Y) \overset{\cong}{\to} \underset{n \in \mathbb{N}}{\oplus} \mathbb{K}(Y)$ for every finite CW-complex Y with trivial action of H. We define $B^{(\infty)}$ as the homotopy colimit of the injections $\begin{cases} B^n & \longrightarrow B^{n+1} \\ (x_1, \ldots, x_n) & \longmapsto (x_1, \ldots, x_n, e) \end{cases}$ for $n \in \mathbb{N}$. An element of $B^{(\infty)}$ may be considered as a family $((x_k)_{k \in \mathbb{N}}, t) \in B^{\mathbb{N}} \times [0, +\infty[$, where $x_k = e$ when $k > [t]$. Let X be a compact space. We define:

$$\alpha_X : \begin{cases} \underset{n \in \mathbb{N}}{\oplus} [X, B] & \longrightarrow [X, B^{(\infty)}] \\ ([f_1], \ldots, [f_n], 0, \ldots) & \longmapsto \left[\begin{cases} X & \to B^n \hookrightarrow B^{(\infty)} \\ x & \mapsto ((f_1(x), \ldots, f_n(x)), n) \end{cases} \right]. \end{cases}$$

Since X is compact, any map $X \to B^{(\infty)}$ may be factored through $X \to B^n \hookrightarrow B^{(\infty)}$ up to homotopy for some $n \in \mathbb{N}$, and similarly for any homotopy $X \times I \to B^{(\infty)}$. We deduce that α_X is a bijection.

Finally, we define a multiplicative structure on $B^{(\infty)}$ by:

$$\begin{cases} B^{(\infty)} \times B^{(\infty)} & \longrightarrow B^{(\infty)} \\ [((x_1, \ldots, x_m, \ldots), t), ((y_1, \ldots, y_n, \ldots), t')] & \longmapsto [(x_1.y_1, \ldots, x_i.y_i, \ldots), \sup(t, t')]. \end{cases}$$

This map is well-defined, continuous, and $(e, 0)$ is a (strict) unit for the H-space $B^{(\infty)}$ (in particular $(e, 0).(e, 0) = (e, 0)$). We easily see that α_X is a homomorphism of abelian monoids. Since this isomorphism is functorial, we deduce that $B^{(\infty)}$ classifies $\mathbb{K}F_H$ on the category of finite CW-complexes with trivial action of H.

• **The case $F = \mathbb{R}$.** For any $j \in J$ and an arbitrary V_j in j, we set $D_j := \mathrm{End}_G(V_j)$. Then D_j is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (see Proposition A.1 of [11]). For $F_1 = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we let J_{F_1} denote the subset of those elements j in J such that $D_j \cong F_1$.

Then, by an argument similar to the one in the complex case, we obtain a natural isomorphism

$$\mathbb{K}O_H(Y) \overset{\cong}{\longrightarrow} \left(\underset{j \in J_{\mathbb{R}}}{\oplus} \mathbb{K}O(Y) \right) \oplus \left(\underset{j \in J_{\mathbb{C}}}{\oplus} \mathbb{K}(Y) \right) \oplus \left(\underset{j \in J_{\mathbb{H}}}{\oplus} \mathbb{K}Sp(Y) \right)$$

on the category of finite CW-complexes with trivial action of H. The proof then runs similarly to the one in the complex case, but this time we consider a classifying space B_{F_1} of $\mathbb{K}F_1(-)$ for each $F_1 = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. $\qquad \square$

We will also need the following theorem: its very technical proof is given in appendix B:

Theorem 4.6. *Let G denote a Lie group, and H be a compact subgroup of G. Then $(\mathrm{Vec}_G^{F,\infty})^H$, $(i\,\mathrm{Vec}_G^{F,\infty})^H$ and $(s\,\mathrm{Vec}_G^{F,\infty})^H$ all have the homotopy type of a CW-complex.*

Proof of Proposition 4.3: Given a compact subgroup H of G, $(\underline{\mathrm{Vec}}_G^{F,\infty}(-))^H$ is a Γ-space. By Theorem 4.6, $(\underline{\mathrm{Vec}}_G^{F,\infty}(\mathbf{1}))^H = (\mathrm{Vec}_G^{F,\infty})^H$ has the homotopy type of a CW-complex. Proposition 3.4.3 shows that

$$\pi_0((\mathrm{Vec}_G^{F,\infty})^H) = [*,(\mathrm{Vec}_G^{F,\infty})^H] = [G/H,\mathrm{Vec}_G^{F,\infty}]_G \cong \mathbb{V}\mathrm{ect}_G^F(G/H) \cong \mathbb{V}\mathrm{ect}_H^F(*) = \mathrm{Rep}_F(H).$$

hence $\mathrm{Rep}_F(H) \cong \pi_0((\mathrm{Vec}_G^{F,\infty})^H)$ (i.e. the monoid of isomorphism classes of finite-dimensional linear representations of F) contains a free and cofinal sub-monoid: the one generated by the irreducible representations of H. Finally, $\Omega B(\underline{\mathrm{Vec}}_G^{F,\infty})^H = (KF_G^{[\infty]})^H$.

We deduce from Proposition 4.1 of [14], applied to the Γ-space $(\underline{\mathrm{Vec}}_G^{F,\infty})^H$, that, for every contravariant functor F from the category of finite CW-complexes to $\mathcal{A}b$ which is classified by an H-space, and every natural transformation $\eta : [-,(\mathrm{Vec}_G^{F,\infty})^H] \longrightarrow F$, there is a unique natural transformation $\eta' : [-,(KF_G^{[\infty]})^H] \longrightarrow F$ which extends η. In particular, any natural transformation $[-,(KF_G^{[\infty]})^H] \longrightarrow [-,(KF_G^{[\infty]})^H]$ under $[-,(\mathrm{Vec}_G^{F,\infty})^H]$ is the identity.

For any finite CW-complex Y with trivial action of G, we deduce from Proposition 3.4.3 that

$$[Y,(\mathrm{Vec}_G^{F,\infty})^H] = [(G/H)\times Y,\mathrm{Vec}_G^{F,\infty}]_G \cong \mathbb{V}\mathrm{ect}_G^F((G/H)\times Y).$$

By Proposition 4.5, $\mathbb{K}F_G((G/H)\times -)$ is classified by an H-space on the category of finite CW-complexes on which G acts trivially. We deduce that there is a unique natural transformation $\delta : [-,(KF_G^{[\infty]})^H] \to \mathbb{K}F_G((G/H)\times -)$ which extends $[-,(\mathrm{Vec}_G^{F,\infty})^H] \longrightarrow \mathbb{K}F_G((G/H)\times -)$.

On the other hand, by the universal property of the Grothendieck construction, for every contravariant functor F from the category of finite CW-complexes to $\mathcal{A}b$, and every natural transformation $\mathbb{V}\mathrm{ect}_G^F((G/H)\times -) \longrightarrow F$, there exists a unique natural transformation $\mathbb{K}F_G((G/H)\times -) \longrightarrow F$ which is compatible with the preceding one. In particular, any natural transformation $\mathbb{K}F_G((G/H)\times -) \longrightarrow \mathbb{K}F_G((G/H)\times -)$ under $\mathbb{V}\mathrm{ect}_G^F((G/H)\times -) \cong [-,(\mathrm{Vec}_G^{F,\infty})^H]$ is the identity. From there, it easily follows that $\gamma_{(G/H)\times Y}$ is an isomorphism for any finite CW-complex Y on which G acts trivially. $\square$

4.4 Alternative classifying spaces

The space $KF_G^{[\infty]}$ is not the only relevant classifying space for the functor $KF_G(-)$ on the category of proper G-CW-complexes. Indeed, $iKF_G^{[\infty]}$ and $sKF_G^{[\infty]}$ (resp. $KF_G^{[m]}$, for $m \in \mathbb{N}^*$) also qualify when G is second countable (resp. discrete).

Proposition 4.7. *Whenever $\pi_0(G)$ is countable (i.e. G is second-countable), the sequence of morphisms of equivariant Γ-spaces $i\underline{\mathrm{Vec}}_G^{F,\infty} \to s\underline{\mathrm{Vec}}_G^{F,\infty} \to \underline{\mathrm{Vec}}_G^{F,\infty}$ induces a sequence of G-weak equivalences $iKF_G^{[\infty]} \to sKF_G^{[\infty]} \to KF_G^{[\infty]}$ hence a sequence of isomorphisms*

$$[X, iKF_G^{[\infty]}]_G \xrightarrow{\cong} [X, sKF_G^{[\infty]}]_G \xrightarrow{\cong} KF_G(X)$$

for every proper G-CW-complex X. It follows that $iKF_G^{[\infty]}$ and $sKF_G^{[\infty]}$ are classifying spaces for $KF_G(-)$ whenever G is second-countable.

Proof. It suffices to check that the maps $(iKF_G^{[\infty]})^H \to (sKF_G^{[\infty]})^H$ and $(sKF_G^{[\infty]})^H \to (KF_G^{[\infty]})^H$ are homotopy equivalences for every compact subgroup H of G. It follows that we only need to make sure that $B(s\,\mathrm{Vec}_G^{F,\infty})^H \to B(\mathrm{Vec}_G^{F,\infty})^H$ and $B(i\,\mathrm{Vec}_G^{F,\infty})^H \to B(s\,\mathrm{Vec}_G^{F,\infty})^H$ are homotopy equivalences. Using statement (ii) in proposition A.1 of [14] and the definition of a $\Gamma - G$-space, we are thus reduced to proving that the maps $(i\,\mathrm{Vec}_G^{F,\infty})^H \to (s\,\mathrm{Vec}_G^{F,\infty})^H \to (\mathrm{Vec}_G^{F,\infty})^H$ are homotopy equivalences. However, by Proposition 3.14, the maps $i\,\mathrm{Vec}_G^{F,\infty} \to s\,\mathrm{Vec}_G^{F,\infty} \to \mathrm{Vec}_G^{F,\infty}$ are G-weak equivalences, hence the maps $(i\,\mathrm{Vec}_G^{F,\infty})^H \to (s\,\mathrm{Vec}_G^{F,\infty})^H \to (\mathrm{Vec}_G^{F,\infty})^H$ are weak equivalences. By Theorem 4.6 and Whitehead's theorem (cf. Theorem 4.5 of [4]), those maps are actually homotopy equivalences. $\square$

Proposition 4.8. *Let G be a discrete group and $m \in \mathbb{N}^*$. Then the morphism $KF_G^{[m]} \to KF_G^{[\infty]}$ in $CG_G^{h\bullet}$ (which is well-defined by Lemma 3.17) is a G-weak equivalence, and, for every proper G-CW-complex, it gives rise to an isomorphism $[X, KF_G^{[m]}]_G \xrightarrow{\cong} KF_G(X)$. Hence $KF_G^{[m]}$ is a classifying space for $KF_G(-)$.*

Proof. We choose a linear injection $\alpha : F^m \hookrightarrow F^{(\infty)}$. It suffices to check that, for every compact subgroup H of G, the map $(\mathrm{Vec}_G^{F,m})^H \to (\mathrm{Vec}_G^{F,\infty})^H$ induced by $(\alpha, \mathrm{id}_{\mathbb{N}})$ is a homotopy equivalence, since $(\underline{\mathrm{Vec}}_G^{F,m})^H$ is a Γ-space. By Theorem 4.6 and Whitehead's theorem, we are reduced to proving that $\mathrm{Vec}_G^{F,m} \to \mathrm{Vec}_G^{F,\infty}$ is a G-weak equivalence, i.e. that

$$[X, \mathrm{Vec}_G^{F,m}]_G \longrightarrow [X, \mathrm{Vec}_G^{F,\infty}]_G$$

is an isomorphism for every proper G-CW-complex X. Since G is discrete, we may apply Proposition 3.4.3 twice to obtain that $[X, \mathrm{Vec}_G^{F,m}]_G \xrightarrow{\cong} \mathbb{Vect}_G^F(X)$ and $[X, \mathrm{Vec}_G^{F,\infty}]_G \xrightarrow{\cong} \mathbb{Vect}_G^F(X)$ are isomorphisms for any proper G-CW-complex X. The result then follows from point (c) of Proposition 3.17. $\square$

4.5 Products in equivariant K-theory

Here, we will follow the ideas from Section 2 of [9]. For every pair (G, H) of Lie groups, we build a pairing $KF_G^{[\infty]} \wedge KF_H^{[\infty]} \to KF_{G \times H}^{[\infty]}$, i.e. a morphism in the

category $\mathrm{CG}^{h\bullet}_{G\times H}[W^{-1}_{G\times H}]$. This pairing is obtained by constructing functorial maps

$$\underline{\mathrm{Vec}}^{F,\infty}_G(S) \times \underline{\mathrm{Vec}}^{F,\infty}_H(T) \to \underline{\mathrm{Vec}}^{F,\infty}_{G\times H}(S \times T)$$

induced by functors

$$\varphi(S)\text{-mod} \times \varphi(T)\text{-mod} \longrightarrow \varphi(S \times T)\text{-mod},$$

where $\varphi = \mathrm{Fib}^{F^{(\infty)}}$. A cautious reader will find a difference in the definition of pairings between ours and that of [9]. In fact, the definition adopted in [9] carries a sign error, and therefore cannot lead to the claimed results. There is another problem in [9]: there was a misinterpretation of some construction of Segal which lead to the false claim that there is a natural *homotopy equivalence* $\Omega BA \to \Omega^2 B^2 A$, when $\underline{A}$ is a good $\Gamma - G$-space. This is fixed in Section C of the appendix.

4.5.1   Hilbert Γ-bundles with a product structure

For any $n \in \mathbb{N}^*$, we define a functor $\pi^{(n)}_\Gamma : \Gamma^n \to \Gamma$ as follows:

$$\pi^{(n)}_\Gamma : \begin{cases} (S_i)_{1\leq i \leq n} & \longmapsto \prod_{i=1}^n S_i \\[2em] (f_i : S_i \to T_i)_{1\leq i \leq n} & \longmapsto \left(\pi^{(n)}_\Gamma((f_i)_{1\leq i \leq n}) : \{(s_i)_{1\leq i \leq n}\} \mapsto \prod_{1\leq i \leq n} f_i(\{s_i\}) \right) \end{cases}$$

Given an list $(x_1, \ldots, x_n)$ of objects of Γ-Fib_F, with $x_i = (S_i, X_i, (p_j)_{j \in S_i})$, set

$$\prod_{i=1}^n x_i := \left(\prod_{i=1}^n S_i, \prod_{i=1}^n X_i, (p_{j_1} \otimes \cdots \otimes p_{j_n})_{j_1 \in S_1, \ldots, j_n \in S_n} \right)$$

For every $n \in \mathbb{N}^*$, these products yield a functor $\pi^{(n)} : (\Gamma\text{-}\mathrm{Fib}_F)^n \longrightarrow \Gamma\text{-}\mathrm{Fib}_F$ defined by

$$\begin{cases} (\varphi_i)_{1\leq i \leq n} & \longmapsto \prod_{i=1}^n \varphi_i \\[2em] ((\gamma_i, f_i, (f^i_t)_{t \in T_i}))_{1\leq i \leq n} & \longmapsto \left(\pi^{(n)}_\Gamma((\gamma_i)_{1\leq i \leq n}), \prod_{i=1}^n f_i, \left(\overset{n}{\underset{i=1}{\otimes}} f^i_{t_i} \right)_{(t_i)_{1\leq i \leq n} \in \prod_{i=1}^n T_i} \right) \end{cases}$$

Let $\varphi : \Gamma \to \Gamma\text{-}\mathrm{Fib}_F$ be a Hilbert Γ-bundle, and $n \geq 2$ be an integer. There are two "natural" functors $\varphi \circ \pi^{(n)}_\Gamma$ and $\pi^{(n)} \circ \varphi^n$ from $\Gamma^{(n)}$ to $\Gamma\text{-}\mathrm{Fib}_F$. The diagram

$$\begin{array}{ccc} \Gamma^n & \overset{\varphi^n}{\longrightarrow} & (\Gamma\text{-}\mathrm{Fib}_F)^n \\ \pi^{(n)}_\Gamma \downarrow & & \downarrow \pi^{(n)} \\ \Gamma & \overset{\varphi}{\longrightarrow} & \Gamma\text{-}\mathrm{Fib}_F \end{array}$$

might not be commutative, which motivates the next definition.

Definition 4.9. If n is an integer greater or equal to 2, and φ a Hilbert Γ-bundle, an n-**fold product structure** on φ is a natural transformation

$$m : \pi^{(n)} \circ \varphi^n \to \varphi \circ \pi_\Gamma^{(n)}$$

such that $\mathcal{O}_\Gamma^F(m) = \mathrm{id}_\Gamma^{(n)}$. A 2-fold product structure is simply called a **product structure**.
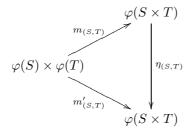
A product structure on a Hilbert Γ-bundle φ is simply the data, for every pair (S, T) of finite sets, of a morphism $\varphi(S) \times \varphi(T) \to \varphi(S \times T)$ in the category Γ-Fib_F, with some additional compatibility conditions.

Definition 4.10. Let n be an integer greater or equal to 2, φ be a Hilbert Γ-bundle, and m and m' be two n-fold product structures on φ.
We say that m **maps to** m' if and only if there exists a natural transformation $\eta : \varphi \circ \pi_\Gamma^{(n)} \to \varphi \circ \pi_\Gamma^{(n)}$ such that $\eta \circ m = m'$.
We say that m and m' are **equivalent** if and only if they belong to the same class for the equivalence relation (on the set of n-fold product structures on φ) generated by the binary relation "map to".

Given two product structures m and m' on a Hilbert Γ-bundle φ, m maps to m' iff we have the data, for every pair of finite sets (S, T), of a morphism $\eta_{(S,T)} : \varphi(S \times T) \to \varphi(S \times T)$ such that the diagram

commutes, with some additional compatibility conditions between the $\eta_{(S,T)}$'s.

Definition 4.11. We define the transposition (or "twist") functor

$$T_\Gamma : \begin{cases} \Gamma \times \Gamma & \longrightarrow \Gamma \times \Gamma \\ (S, T) & \longmapsto (T, S) \\ (\gamma_1, \gamma_2) & \longmapsto (\gamma_2, \gamma_1). \end{cases}$$

Definition 4.12. Let Φ and Ψ be two functors from Γ^2 to Γ-Fib_F, and $f : \Phi \to \Psi$ be a natural transformation such that $\mathcal{O}_\Gamma^F(f_{(S,T)}) = \mathrm{id}_{T \times S}$ for every pair (S, T) of finite sets.
We define Tf as a natural transformation between functors from Γ^2 to Γ-Fib_F as follows: for every pair (S, T) of finite sets, if $f_{(S,T)} = (\mathrm{id}_{T \times S}, g, (g_{t,s})_{(t,s) \in T \times S})$, then

$$(Tf)_{(S,T)} := (\mathrm{id}_{S \times T}, g, (g_{t,s})_{(s,t) \in S \times T}).$$

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If φ is a Hilbert Γ-bundle, and m is a product structure on φ, one can easily check that $T(m \circ T_\Gamma)$ is another product structure on φ. Moreover, both $m \circ (m \times \mathrm{id})$ and $m \circ (\mathrm{id} \times m)$ are 3-fold product structures on φ.

Definition 4.13. Let φ be a Hilbert Γ-bundle, and m be a product structure on φ. Then m is called a **good** product structure when the product structures m and $T(m \circ T_\Gamma)$ are equivalent and the 3-fold product structures $m \circ (m \times \mathrm{id})$ and $m \circ (\mathrm{id} \times m)$ are equivalent.

Remark 12. A product structure is good when it is commutative "up to natural transformations", and associative "up to natural transformations".

4.5.2 Products on KF_G^φ

Here φ will denote a Hilbert Γ-bundle, n an integer greater or equal to 2, and m an n-fold product structure on φ. For any Lie group G, we set

$$KF_G^\varphi := \Omega B \operatorname{Vec}_G^\varphi .$$

For every n-tuple of finite sets $(S_i)_{1 \le i \le n}$, m induces a functor

$$m((S_i)_{1 \le i \le n})\text{-mod} : \left(\prod_{i=1}^n \varphi(S_i) \right)\text{-mod} \longrightarrow \varphi \left(\prod_{i=1}^n S_i \right)\text{-mod defined by}$$

$$m((S_i)_{1 \le i \le n})\text{-mod} : \begin{cases} (x_i)_{1 \le i \le n} & \longmapsto m_{(S_i)_{1 \le i \le n}}((x_i)_{1 \le i \le n}) \\ (x_i \overset{f_{s_i}}{\to} x_i')_{(s_i)_{1 \le i \le n} \in \prod\limits_{i=1}^n S_i} & \longmapsto \left(m_{(S_i)_{1 \le i \le n}} \circ \left(\overset{n}{\underset{i=1}{\otimes}} f_{s_i} \right)_{(s_i)_{1 \le i \le n} \in \prod\limits_{i=1}^n S_i} \circ m_{(S_i)_{1 \le i \le n}}^{-1} \right) \end{cases}.$$

Let $(G_i)_{1 \le i \le n}$ be an n-tuple of Lie groups. We set $G := \prod\limits_{i=1}^n G_i$. The functors $m((S_i)_{1 \le i \le n})$-mod induce a continuous map

$$B^n m : \prod_{i=1}^n B \operatorname{Vec}_{G_i}^\varphi \longrightarrow B^n \operatorname{Vec}_G^\varphi,$$

where $B^n \operatorname{Vec}_G^\varphi$ denotes the n-fold realization of the Γ-space $\underline{\operatorname{Vec}}_G^\varphi$ (cf. [14] §1). This yields a pointed G-map $\bigwedge\limits_{i=1}^n B \operatorname{Vec}_{G_i}^\varphi \longrightarrow B^n \operatorname{Vec}_G^\varphi / B_0^n \operatorname{Vec}_G^\varphi$ and, furthermore, a morphism $\bigwedge\limits_{i=1}^n B \operatorname{Vec}_{G_i}^\varphi \longrightarrow B^n \operatorname{Vec}_G^\varphi$ in the category $CG_G^{h \bullet}$. Our first result is the following key lemma:

Lemma 4.14. *Let m and m' be two product structures of order n on a Hilbert Γ-bundle φ. If m and m' are equivalent, then they induce the same morphism $\bigwedge\limits_{i=1}^n B \operatorname{Vec}_{G_i}^\varphi \longrightarrow B^n \operatorname{Vec}_G^\varphi$ in the category $CG_G^{h \bullet}$.*

Proof. It suffices to prove the result when m maps to m'. However, any natural transformation η from m to m' induces a pointed equivariant homotopy from $B^n m$ to $B^n m'$, by a result that is essentially similar to Lemma 3.16. $\qquad\square$

The previous morphism yields a morphism

$$m^* : \Omega^n\left(\bigwedge_{i=1}^n B\operatorname{Vec}_{G_i}^\varphi\right) \longrightarrow \Omega^n B^n \operatorname{Vec}_G^\varphi$$

in the category $CG_G^{h\bullet}$. We also have a pointed G-map:

$$\prod_{i=1}^n \Omega B\operatorname{Vec}_{G_i}^\varphi \longrightarrow \Omega^n\left(\bigwedge_{i=1}^n B\operatorname{Vec}_{G_i}^\varphi\right)$$

which yields a morphism in the category $CG_G^{h\bullet}$:

$$\bigwedge_{i=1}^n KF_{G_i}^\varphi \longrightarrow \Omega^n\left(\bigwedge_{i=1}^n B\operatorname{Vec}_{G_i}^\varphi\right).$$

Composing this last morphism with the above one yields a morphism in $CG_G^{h\bullet}$:

$$\bigwedge_{i=1}^n KF_{G_i}^\varphi \longrightarrow \Omega^n B^n \operatorname{Vec}_G^\varphi.$$

Composing it with the inverse of $-(i_{n-1}^0 \circ \cdots \circ i_1^0) : \Omega B\operatorname{Vec}_G^\varphi \longrightarrow \Omega^n B^n \operatorname{Vec}_G^\varphi$ in the category $CG_G^{h\bullet}[W_G^{-1}]$ (cf. the definition of the i_j^k's in Section C of the appendix), finally yields a morphism

$$\bigwedge_{i=1}^n KF_{G_i}^\varphi \longrightarrow KF_G^\varphi$$

in the category $CG_G^{h\bullet}[W_G^{-1}]$. By Lemma 4.14, this morphism only depends on the isomorphism class of the chosen n-fold product structure m. We may rewrite the previous morphism as the composite morphism:

$$\bigwedge_{i=1}^n KF_{G_i}^\varphi \longrightarrow \Omega^n\left(\bigwedge_{i=1}^n B\operatorname{Vec}_{G_i}^\varphi\right) \xrightarrow{m^*} \Omega^n B^n \operatorname{Vec}_G^\varphi \xleftarrow{-(i_{n-1}^0 \circ \cdots \circ i_1^0)} KF_G^\varphi$$

Remarks 13.   (i) By Corollary C.2 in the appendix, we would have obtained the same morphism by using the inverse of $-(i_{n-1}^{k_{n-1}} \circ \cdots \circ i_1^{k_1})$ in $CG_G^{h\bullet}[W_G^{-1}]$ for any $(k_1, \ldots, k_{n-1}) \in \prod_{j=0}^{n-2}[j]$, instead of using the inverse of $-(i_{n-1}^0 \circ \cdots \circ i_1^0)$.

(ii) This construction is different from that in [9], where instead of the inverse of $-(i_{n-1}^0 \circ \cdots \circ i_1^0)$, the "opposite map" is used (in the sense of inversion of the order of looping). The construction of [9] however leads to false claims on the ring structure of $KF_G^*(-)$.

(iii) The morphism $\overset{n}{\underset{i=1}{\wedge}} KF^{\varphi}_{G_i} \longrightarrow KF^{\varphi}_G$ we have just constructed is clearly natural with respect to the Lie groups $G_1, \ldots, G_n$.

Let now m be a product structure on φ, and G and H be two Lie groups. Composing with the above morphism, yields a map

$$[X, KF^{\varphi}_G]^{\bullet}_G \times [Y, KF^{\varphi}_H]^{\bullet}_H \longrightarrow [X \wedge Y, KF^{\varphi}_G \wedge KF^{\varphi}_H]^{\bullet}_{G \times H} \longrightarrow [X \wedge Y, KF^{\varphi}_{G \times H}]^{\bullet}_{G \times H}$$

for every pointed proper G-space X and every pointed proper H-space Y.

4.5.3 Properties of the product given by a good product structure

Here φ will denote a Hilbert Γ-bundle, and m a good product structure on φ. Let G and H be two Lie groups. The following results are easily shown to follow from the definition of a good product structure (see [9] for details):

Proposition 4.15. *The product map* $[X, KF^{\varphi}_G]^{\bullet}_G \times [Y, KF^{\varphi}_H]^{\bullet}_H \longrightarrow [X \wedge Y, KF^{\varphi}_{G \times H}]^{\bullet}_{G \times H}$ *induced by m is bilinear, functorial with respect to X and Y on the one hand, and with respect to G and H on the other hand.*

Proposition 4.16. *The product induced by m is commutative, i.e. for any pointed proper G-CW-complex X and any pointed proper H-CW-complex Y, the following square is commutative*

$$
\begin{array}{ccc}
[X, KF^{\varphi}_G]^{\bullet}_G \times [Y, KF^{\varphi}_H]^{\bullet}_H & \longrightarrow & [X \wedge Y, KF^{\varphi}_{G \times H}]^{\bullet}_{G \times H} \\
i \downarrow & & j^* \downarrow \\
[Y, KF^{\varphi}_H]^{\bullet}_H \times [X, KF^{\varphi}_G]^{\bullet}_G & \longrightarrow & [Y \wedge X, KF^{\varphi}_{H \times G}]^{\bullet}_{H \times G}
\end{array}
$$

where i is the transposition of factors, and j is induced by the transposition map $Y \wedge X \overset{\cong}{\longrightarrow} X \wedge Y$ and by the map $KF^{\varphi}_{G \times H} \to KF^{\varphi}_{H \times G}$, which is itself induced by the functor $\mathcal{E}(H \times G) \to \mathcal{E}(G \times H)$ associated to the transposition map $H \times G \overset{\cong}{\longrightarrow} G \times H$.

Proposition 4.17. *The product associated to m is associative, i.e. for any triple (G_1, G_2, G_3) of Lie groups, for any triple (X_1, X_2, X_3) such that X_i is a pointed proper G_i-CW-complex for every $i \in \{1, 2, 3\}$, the square*

$$
\begin{array}{ccc}
[X_1, KF^{\varphi}_{G_1}]^{\bullet}_{G_1} \times [X_2, KF^{\varphi}_{G_2}]^{\bullet}_{G_2} \times [X_3, KF^{\varphi}_{G_3}]^{\bullet}_{G_3} & \overset{a}{\longrightarrow} & [X_1 \wedge X_2, KF^{\varphi}_{G_1 \times G_2}]^{\bullet}_{G_1 \times G_2} \times [X_3, KF^{\varphi}_{G_3}]^{\bullet}_{G_3} \\
b \downarrow & & c \downarrow \\
[X_1, KF^{\varphi}_{G_1}]^{\bullet}_{G_1} \times [X_2 \wedge X_3, KF^{\varphi}_{G_2 \times G_3}]^{\bullet}_{G_2 \times G_3} & \overset{d}{\longrightarrow} & [X_1 \wedge X_2 \wedge X_3, KF^{\varphi}_{G_1 \times G_2 \times G_3}]^{\bullet}_{G_1 \times G_2 \times G_3}
\end{array}
$$

induced by the good product structure m is commutative.

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4.5.4 Typical product structures on $\mathrm{Fib}^{F^{(\infty)}}$

Let n be an integer greater than or equal to 2, and $\alpha : \mathbb{N}^n \overset{\cong}{\Rightarrow} \mathbb{N}$ be a bijection. Identifying the canonical basis of $F^{(\infty)}$ with $\mathbb{N}$, we use α to obtain an isomorphism $\alpha_F : \overset{n}{\underset{i=1}{\otimes}} F^{(\infty)} \overset{\cong}{\Rightarrow} F^{(\infty)}$ which is bicontinuous with respect to the limit topology for the inclusion of finite-dimensional subspaces. It follows that we obtain a bicontinuous isomorphism

$$\alpha^F : \begin{cases} \overset{n}{\underset{i=1}{\bigotimes}} \left[\underset{k \in \mathbb{N}}{\oplus} (F^{(\infty)})^{\{k\}} \right] & \longrightarrow \underset{k \in \mathbb{N}}{\oplus} (F^{(\infty)})^{\{k\}} \\ x_1 \otimes \cdots \otimes x_n \in (F^{(\infty)})^{\{i_1\}} \otimes \cdots \otimes (F^{(\infty)})^{\{i_n\}} & \longmapsto \alpha_F(x_1 \otimes \cdots \otimes x_n) \in (F^{(\infty)})^{\{\alpha(i_1,\ldots,i_n)\}}, \end{cases}$$

where, for every $k \in \mathbb{N}$, $(F^{(\infty)})^{\{k\}}$ denotes the space of maps $\{k\} \to F^{(\infty)}$ (seen as a subspace of the space of maps from $\mathbb{N}$ to $F^{(\infty)}$).

For every n-tuple $(A_1, \ldots, A_n)$ of finite subsets of $\mathbb{N}$, the map α thus induces a cartesian square

$$\begin{array}{ccc}
E_{A_1}(F^{(\infty)}) \otimes \cdots \otimes E_{A_n}(F^{(\infty)}) & \longrightarrow & E_{\alpha(A_1 \times \cdots \times A_n)}(F^{(\infty)}) \\
\downarrow & & \downarrow \\
G_{A_1}(F^{(\infty)}) \times \cdots \times G_{A_n}(F^{(\infty)}) & \longrightarrow & G_{\alpha(A_1 \times \cdots \times A_n)}(F^{(\infty)})
\end{array}$$

in which the top horizontal morphism is defined by

$$\begin{cases} E_{A_1}(F^{(\infty)}) \otimes \cdots \otimes E_{A_n}(F^{(\infty)}) & \longrightarrow E_{\alpha(A_1 \times \cdots \times A_n)}(F^{(\infty)}) \\ (x_1, V_{x_1}) \otimes \cdots \otimes (x_n, V_{x_n}) & \longmapsto (\alpha^F(x_1 \otimes \cdots \otimes x_n), \alpha^F(V_{x_1} \otimes \cdots \otimes V_{x_n})). \end{cases}$$

Those cartesian squares define strong morphisms of Hilbert bundles, and thus induce a natural transformation:

$$m_\alpha : \pi^{(n)} \circ \left(\mathrm{Fib}^{F^{(\infty)}} \right)^n \longrightarrow \mathrm{Fib}^{F^{(\infty)}} \circ \pi_\Gamma^{(n)},$$

hence m_α is an n-fold product structure on $\mathrm{Fib}^{F^{(\infty)}}$.

Proposition 4.18. *The equivalence class of the n-fold product structure m_α on $\mathrm{Fib}^{F^{(\infty)}}$ does not depend on the choice of the bijection $\alpha : \mathbb{N}^n \overset{\cong}{\Rightarrow} \mathbb{N}$.*

Proof. Let α and β be two isomorphisms $\mathbb{N}^n \overset{\cong}{\Rightarrow} \mathbb{N}$. Then $\gamma := \beta \circ \alpha^{-1}$ is a permutation of $\mathbb{N}$, and it induces an automorphism $\gamma' : F^{(\infty)} \overset{\cong}{\longrightarrow} F^{(\infty)}$. We thus obtain a natural transformation $(\gamma', \gamma)^* : \mathrm{Fib}^{F^{(\infty)}} \to \mathrm{Fib}^{F^{(\infty)}}$ (cf. Section 3.4.5) such that $m_\beta = (\gamma', \gamma)^*_{\pi_\Gamma^{(n)}} \circ m_\alpha$. This proves that m_β is equivalent to m_α. $\qquad\square$

Proposition 4.19. *For any bijection $\alpha : \mathbb{N} \times \mathbb{N} \overset{\cong}{\longrightarrow} \mathbb{N}$, the product structure m_α on $\mathrm{Fib}^{F^{(\infty)}}$ is good.*

Proof. Let $T_\mathbb{N}$ be the transposition of factors of $\mathbb{N}^2$. Then $T(m_\alpha \circ T_\Gamma) = m_{\alpha \circ T_\mathbb{N}}$. We deduce from Proposition 4.18 that m_α and $T(m_\alpha \circ T_\Gamma)$ are equivalent. Moreover $m_\alpha \circ (m_\alpha \times \mathrm{id}) = m_{\alpha \circ (\alpha \times \mathrm{id})}$ and $m_\alpha \circ (\mathrm{id} \times m_\alpha) = m_{\alpha \circ (\mathrm{id} \times \alpha)}$ We deduce from Proposition 4.18 that $m_\alpha \circ (m_\alpha \times \mathrm{id})$ and $m_\alpha \circ (\mathrm{id} \times m_\alpha)$ are equivalent. $\quad\square$

4.5.5 The product maps in equivariant K-theory

Let G and H be two Lie groups. If we choose a bijection $\alpha : \mathbb{N} \times \mathbb{N} \overset{\cong}{\Rightarrow} \mathbb{N}$, we obtain a good product structure m_α on $\mathrm{Fib}^{F^{(\infty)}}$ which yields a morphism $KF_G^{[\infty]} \wedge KF_H^{[\infty]} \to KF_{G \times H}^{[\infty]}$ in the category $CG_{G \times H}^{h\bullet}[W_{G \times H}^{-1}]$. Proposition 4.18 and Lemma 4.14 show this is independent from the choice of α.

Given a proper G-CW-complex X and a proper H-CW-complex Y, we then obtain a map

$$m_\alpha^* : KF_G(X) \times KF_H(Y) \longrightarrow [X^+, KF_G^{[\infty]}]_G^\bullet \times [Y^+, KF_G^{[\infty]}]_H^\bullet \longrightarrow [X^+ \wedge Y^+, KF_{G \times H}^{[\infty]}]_{G \times H}^\bullet$$
$$= KF_{G \times H}(X \times Y).$$

For any proper G-CW-pair (X, A) and any proper H-CW-pair (Y, B), a similar procedure yields, for every pair $(m, n) \in \mathbb{N}^2$, a map

$$KF_G^{-m}(X, A) \times KF_G^{-n}(Y, B) \longrightarrow KF_{G \times H}(\Sigma^m(X/A) \wedge \Sigma^n(Y/B), *).$$

These are our product maps in equivariant K-theory. Since m_α is a good product structure, Proposition 4.16 and 4.17 show that they are "commutative" and "associative". Just like in [9], one then proves:

Proposition 4.20. *Let G and H be two Lie groups, X be a proper G-CW-complex and Y be a proper H-CW-complex. Then the square*

$$
\begin{array}{ccc}
\mathbb{K}F_G(X) \times \mathbb{K}F_H(Y) & \xrightarrow{\gamma_X \times \gamma_Y} & KF_G(X) \times KF_H(Y) \\
\otimes \downarrow & & m_\alpha^* \downarrow \\
\mathbb{K}F_{G \times H}(X \times Y) & \xrightarrow{\gamma_{X \times Y}} & KF_{G \times H}(X \times Y)
\end{array}
$$

is commutative.

4.6 Bott periodicity and the extension to positive degrees

Assume $F = \mathbb{C}$. By the same method as in [9], the Bott homomorphism

$$\beta_{X,A}^{-n} : KF_G^{-n}(X, A) \longrightarrow KF_G^{-(n+2)}(X, A)$$

is defined for every proper G-CW pair and shown to be an isomorphism. In our case, this uses Proposition 4.3 and the fact that Bott periodicity holds for equivariant K-theory with compact groups (cf. [13]).

For any proper G-CW-complex X, a structure of graded ring may also be defined on

$$KF_G^*(X) := \overset{+\infty}{\underset{n=0}{\oplus}} KF_G^{-n}(X)$$

using the good product structure derived from m_α. Following the arguments from [9], this is easily shown to satisfy the following properties:

Theorem 4.21. *Let G be a Lie group, and X be a proper G-CW-complex. Then:*

(i) *The Bott homomorphisms induce an isomorphism*

$$\beta : KF_G^*(X) \to KF_G^*(X)$$

which is linear for the product of the graded ring $KF_G^*(X)$ (on both sides).

(ii) *The map $\gamma_X : \mathbb{K}_G(X) \to KF_G(X)$ is a ring homomorphism.*

Corollary 4.22. *The cohomology theory $KF_G^*(-)$ may be extended to positive degrees on the category of G-CW-pairs, so that we obtain a good equivariant cohomology theory with a graded ring structure and Bott periodicity.*

Remarks 14.   • The construction of the ring structure and of Bott homomorphisms may also carried out in the case $F = \mathbb{R}$, with the usual modifications.

• All the previous constructions may also be carried out with $iKF_G^{[\infty]}$ and $sKF_G^{[\infty]}$, and this easily yields the same product structure and the same Bott homomorphisms when $\pi_0(G)$ is countable.

5   Connection with other equivariant K-theories

5.1   A connection with Segal's equivariant K-theory

Let G be a Lie group. We have constructed a natural transformation

$$\gamma : \mathbb{K}F_G^*(-) \longrightarrow KF_G^*(-)$$

on the category of G-CW-pairs. Recall that, for every non-negative integer n, every compact subgroup H of G, and every finite complex Y on which G acts trivially,

$$\gamma_{(G/H) \times Y}^{-n} : \mathbb{K}F_G^{-n}((G/H) \times Y) \overset{\cong}{\longrightarrow} KF_G^{-n}((G/H) \times Y)$$

is an isomorphism. Also that $\gamma_X : \mathbb{K}F_G(X) \longrightarrow KF_G(X)$ is a ring homomorphism for every G-CW-complex X.

We now assume that $\mathbb{K}F_G^*(-)$ is a good equivariant cohomology theory on the category of finite proper G-CW-pairs: we know this is true in the case G is a compact Lie group or a discrete group (cf. [8]). In this case, it is easy to check that the natural transformation γ is compatible with the boundary maps

in the respective long exact sequences of a pair for the equivariant cohomology theories $\mathbb{K}F_G^*(-)$ and $KF_G^*(-)$. It follows that the natural transformation γ is compatible both with the long exact sequences of a pair and the long exact sequences of Mayer-Vietoris. As a consequence, we have:

Proposition 5.1. *If $\mathbb{K}F_G^*(-)$ is a good equivariant cohomology theory on the category of finite proper G-CW-pairs, then, for any finite proper G-CW-pair (X, A) and every $n \in \mathbb{N}$, $\gamma_{X,A}^{-n}$ is an isomorphism.*

Proof. By compatibility of γ with the long exact sequence of a pair, and by the five lemma, it suffices to prove that γ_X^{-n} is an isomorphism for every proper finite G-CW-complex X and every non-negative integer n. Let X be such a proper finite G-CW-complex, and $n \in \mathbb{N}$. We prove that γ_X^{-n} is an isomorphism by a double induction process on the dimension of X and the number of cells in a given dimension: this is done by considering the long exact sequences of Mayer-Vietoris associated to push-out squares of the form

$$
\begin{array}{ccc}
G/H \times S^{m-1} & \longrightarrow & G/H \times D^m \\
{\scriptstyle\psi}\downarrow & & \downarrow \\
Y & \longrightarrow & (G/H \times D^m) \cup_\psi Y,
\end{array}
$$

and then by using the compatibility of γ with those sequences, together with the result of Corollary 4.4 and the five lemma. $\qquad\square$

With the same arguments as in Section 2 of [9], one also obtains:

Proposition 5.2. *Assume $\mathbb{K}F_G^*(-)$ is a good equivariant cohomology theory on the category of finite proper G-CW-pairs. Then, for any finite proper G-CW-complex X, the homomorphism*

$$
\gamma_X^* : \overset{+\infty}{\underset{n=0}{\oplus}} \mathbb{K}F_G^{-n}(X) \longrightarrow \overset{+\infty}{\underset{n=0}{\oplus}} KF_G^{-n}(X)
$$

is a ring isomorphism which commutes with the Bott homomorphisms.

5.2 A connection with the K-theory of Lück and Oliver

5.2.1 A natural transformation $KF_G^*(-) \to K'F_G^*(-)$

Here, G will be a discrete group. For any G-CW-pair (X, A) and any integer n, we denote by $K'F_G^n(X, A)$ the Lück-Oliver equivariant K-theory of the pair (X, A) in degree n as defined in [9].

We define the category $\Gamma\text{-Fib}_F^*$ as follows:

- An object of $\Gamma\text{-Fib}_F^*$ consists of a finite set S, of a locally-countable CW-complex X, and, for every $s \in S$, of a finite-dimensional vector bundle (with underlying field F): $p_s : E_s \to X$.

- A morphism $f : (S, X, (p_s)_{s \in S}) \longrightarrow (T, Y, (q_t)_{t \in T})$ consists of a morphism $\gamma : T \to S$ in the category Γ, of a continuous map $\bar{f} : X \to Y$, and for every $t \in T$, of a strong morphism of vector bundles:

$$
\begin{array}{ccc}
\underset{s \in \gamma(t)}{\oplus} E_t & \xrightarrow{\ f_t\ } & E'_t \\
\underset{s \in \gamma(t)}{\oplus} p_s \downarrow & & \downarrow q_t \\
X & \xrightarrow[\ \bar{f}\ ]{} & Y.
\end{array}
$$

The composition of morphisms is defined as in Γ-Fib_F.

A functor $\mathcal{O}_\Gamma^{F*} : \Gamma$-$\mathrm{Fib}_F^* \to \Gamma$ is defined as follows:

$$
\begin{cases}
(S, X, (p_s)_{s \in S}) & \longmapsto S \\
f : (S, X, (p_s)_{s \in S}) \to (T, Y, (q_t)_{t \in T}) & \longmapsto (\gamma : T \to S).
\end{cases}
$$

As in the case of Hilbert Γ-bundles, we may define the sum in Γ-Fib_F^*, and the functors -mod : Γ-$\mathrm{Fib}_F^* \to \mathrm{kCat}$ and -Bdl : Γ-$\mathrm{Fib}_F^* \to \mathrm{kCat}$. Statement (i) of Proposition 3.1 is then true in the case of Γ-Fib_F^*, and so are Proposition 3.2 and Corollary 3.3.

A **Γ-vector bundle** is a contravariant functor $\varphi : \Gamma \to \Gamma$-$\mathrm{Fib}_F^*$ satisfying:

(i) $\mathcal{O}_\Gamma^{F*}(\varphi) = \mathrm{id}_\Gamma$;

(ii) $\varphi(\mathbf{0}) = (\mathbf{0}, *, \emptyset)$;

(iii) $\forall n \in \mathbb{N}^*, \exists f_n : n.\varphi(\mathbf{1}) \to \varphi(\mathbf{n})$ such that $\mathcal{O}_\Gamma^{F*}(f_n) = \mathrm{id}_\mathbf{n}$.

Moreover, if $F = \mathbb{R}$ or $\mathbb{C}$, we have a natural functor Γ-$\mathrm{Fib}_F \to \Gamma$-$\mathrm{Fib}_F^*$ obtained by forgetting the Hilbert structure on the fibers. Any Hilbert Γ-bundle may thus be seen as a Γ-vector bundle. The two -mod constructions are identical.

We define a functor $\psi : \Gamma \longrightarrow \Gamma$-$\mathrm{Fib}_F^*$ in the following way: for any finite set S, we let X_S denote the subset of $\left(\overset{+\infty}{\underset{k=0}{\cup}} \mathrm{sub}_k(F^{(\infty)}) \right)^S$ consisting of those families indexed over S whose factors are in direct sum, and we equip X_S with the discrete topology. For any $s \in S$, we let $p_s^S : E_s^S \to X_S$ denote the canonical vector bundle over X_S whose fiber over any $(x_t)_{t \in S}$ is x_s. We finally set

$$
\psi(S) := (S, X_S, (p_s^S)_{s \in S}).
$$

For every morphism $f : S \to T$ in Γ, the direct sum of subspaces gives rise to a morphism $\psi(f) : \psi(T) \to \psi(S)$ in Γ-Fib_F^*. It is then easy to check that ψ is a Γ-vector bundle.

We observe that the space KF_G defined by Lück and Oliver in [9] is simply KF_G^ψ as defined in the beginning of Section 4.5.2. We now wish to relate KF_G^ψ to $KF_G^{[\infty]}$.

For any finite set S, the underlying space of $\mathrm{Fib}^F(S)$ is the set of S-tuples of subspaces of $\overset{+\infty}{\underset{n=0}{\cup}} F^{[n]}$ that are in direct sum and have a basis consisting of vectors of the canonical basis of $\overset{+\infty}{\underset{n=0}{\cup}} F^{[n]}$. Indeed, for any finite subset A of $\mathbb{N}$, F^A is the only $(\#A)$-dimensional subspace of itself. The isomorphism

$$\begin{cases} \overset{+\infty}{\underset{n=0}{\cup}} F^{[n]} & \overset{\cong}{\longrightarrow} F^{(\infty)} \\ e_n & \longmapsto e_{n+1} \end{cases}$$

yields a canonical injection of $\mathrm{Fib}^F(S)$ into X_S. These injections yield a natural transformation $\varepsilon : \mathrm{Fib}^F \longrightarrow \psi$ between Γ-vector bundles. Then ε gives rise to a morphism of equivariant Γ-spaces:

$$\varepsilon^* : \underline{\mathrm{Vec}}_G^{F,1} \longrightarrow \underline{\mathrm{Vec}}_G^{\psi}.$$

We claim that ε_1^* is an equivariant homotopy equivalence. To see this, we choose, for every non-negative integer n and every n-dimensional subspace E of $F^{(\infty)}$, an isomorphism $E \overset{\cong}{\longrightarrow} F^{[n-1]}$. Those choices yield a strong morphism of vector bundles

$$\begin{array}{ccc} E_1^{\mathbf{1}} & \longrightarrow & \underset{f \in \Gamma(\mathbf{1})}{\coprod} E_{f(1)}(F) \\ p_1^{\mathbf{1}} \downarrow & & \downarrow \\ X_{\mathbf{1}} & \longrightarrow & \underset{f \in \Gamma(\mathbf{1})}{\coprod} B_{f(1)}(F). \end{array}$$

Using results that are essentially similar to the ones of Proposition 3.2 and Lemma 3.16 (adapted to the case of Γ-vector bundles), we conclude that the natural transformation $\varepsilon : \mathrm{Fib}^F \to \psi$ induces an equivariant homotopy equivalence

$$\varepsilon_{\mathbf{1}}^* : \mathrm{Vec}_G^{F,1} \overset{\cong}{\longrightarrow} \mathrm{Vec}_G^{\psi}.$$

Since $\underline{\mathrm{Vec}}_G^{F,1}$ and $\underline{\mathrm{Vec}}_G^{\psi}$ are $\Gamma - G$-spaces, we deduce that $\varepsilon_S^* : \underline{\mathrm{Vec}}_G^{F,1}(S) \overset{\cong}{\longrightarrow} \underline{\mathrm{Vec}}_G^{\psi}(S)$ is an equivariant homotopy equivalence for every finite set S.

We finally deduce from the previous discussion that ε^* induces a G-weak equivalence

$$\varepsilon^* : KF_G^{[1]} \longrightarrow KF_G.$$

Since G is discrete, Proposition 4.8 applied to $m = 1$ shows that the canonical map $KF_G^{[1]} \longrightarrow KF_G^{[\infty]}$ is a G-weak equivalence. By composing its inverse with $\varepsilon^* : KF_G^{[1]} \longrightarrow KF_G$, we recover an isomorphism $KF_G^{[\infty]} \longrightarrow KF_G$ in the category $CG_G^{h\bullet}[W_G^{-1}]$.

For every pointed proper G-CW-complex X, this morphism induces a group isomorphism

$$[X, KF_G^{[\infty]}]_G^{\bullet} \overset{\cong}{\longrightarrow} [X, KF_G]_G^{\bullet}.$$

For every $n \in \mathbb{N}$ and every proper G-CW-pair (X, A), we thus obtain a group isomorphism

$$\varepsilon_{X,A}^{-n} : KF_G^{-n}(X, A) \xrightarrow{\cong} K'F_G^{-n}(X, A).$$

This defines a bijective natural transformation from our equivariant K-theory to the one of Lück and Oliver.

5.2.2   The compatibility of ε with products

We now check that the natural transformation ε is compatible with products and Bott homomorphisms. It actually suffices to prove that it is compatible with exterior products.

We let $\alpha : \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \mathbb{N}$ be a bijection such that $\alpha(0,0) = 0$. We also let

$$\beta : \begin{cases} F & \longrightarrow F^{(\infty)} \\ x & \longmapsto (x, 0, 0, \dots) \end{cases} \text{ denote the canonical injection.}$$

The embedding β helps us see Fib^F as a sub-Hilbert Γ-bundle of $\mathrm{Fib}^{F^{(\infty)}}$. Since $\alpha(0,0) = 0$, we have $\alpha_F(e_1, e_1) = e_1$, and it follows that the good product structure m_α on $\mathrm{Fib}^{F^{(\infty)}}$ induces a good product structure m'_α on Fib^F. Moreover, for any pair of Lie groups (G, H), the square

$$\begin{array}{ccc}
KF_G^{[1]} \wedge KF_H^{[1]} & \longrightarrow & KF_G^{[\infty]} \wedge KF_H^{[\infty]} \\
\downarrow{\scriptstyle (m'_\alpha)^*} & & \downarrow{\scriptstyle m_\alpha^*} \\
KF_{G \times H}^{[1]} & \longrightarrow & KF_{G \times H}^{[\infty]}
\end{array}$$

is commutative in $CG_{G \times H}^{h\bullet}[W_{G \times H}^{-1}]$.

If we assume that G and H are both discrete, then the square

$$\begin{array}{ccc}
KF_G^{[1]} \wedge KF_H^{[1]} & \longrightarrow & KF_G \wedge KF_H \\
\downarrow{\scriptstyle (m'_\alpha)^*} & & \downarrow{\scriptstyle m_{(\alpha_F)}^*} \\
KF_{G \times H}^{[1]} & \longrightarrow & KF_{G \times H}
\end{array}$$

is also commutative in $CG_G^{h\bullet}[W_{G \times H}^{-1}]$, and this follows from our definition of products and the one in Section 2 of [9].

We deduce from the previous two commutative squares that $\varepsilon^* : KF_G^*(-) \longrightarrow K'F_G^*(-)$ is compatible with exterior products. In particular, it is compatible with Bott homomorphisms and it is therefore possible to extend ε to positive degrees. We finally obtain a ring isomorphism

$$\varepsilon_X^* : KF_G^*(X) \xrightarrow{\cong} K'F_G^*(X)$$

for any discrete group G and any proper G-CW-complex X.

We conclude that our equivariant K-theory coincides with the one of Lück and Oliver in the case of discrete groups.

Remark 15. The reason our construction is much more complicated than the one of Lück and Oliver is that the latter fails in the general case of a Lie group. For example, $\mathrm{Vec}_{S^1}^{F,1}$ is not a classifying space for $\mathbb{V}\mathrm{ect}_{S^1}^{\mathbb{R}}(-)$ on the category of S^1-CW-complexes, because the fixed point set $(\widetilde{\mathrm{Vec}}_G^{F,1})^{\{(-1,-1),(1,1)\}}$ for the subgroup $\{(-1,-1),(1,1)\} \subset S^1 \times \mathrm{GL}_1(\mathbb{R})$ is empty. The additional complexity we introduced was there so that Proposition 2.8 would hold.

A On some fiber bundles

Our aim here is to give a complete proof of Theorem 2.6, which we now restate:

Theorem A.1. *Let X be a locally-countable CW-complex, $\varphi : E \to X$ be an n-dimensional vector bundle over X, and G be a Lie group. Then:*

(i) $\widetilde{\mathrm{Vec}}_G^{\varphi} \longrightarrow \mathrm{Vec}_G^{\varphi}$ *is a $(G, \mathrm{GL}_n(F))$-principal bundle;*

(ii) $E\,\mathrm{Vec}_G^{\varphi} \longrightarrow \mathrm{Vec}_G^{\varphi}$ *is an n-dimensional G-vector bundle;*

(iii) The canonical map

$$\widetilde{\mathrm{Vec}}_G^{\varphi} \times_{\mathrm{GL}_n(F)} F^n \longrightarrow E\,\mathrm{Vec}_G^{\varphi}$$

is an isomorphism of G-vector bundles over Vec_G^{φ}.

There is a free right-action of $\mathrm{GL}_n(F)$ on φ-frame, and this induces a free action of $\mathrm{GL}_n(F)$ on $|\,\mathrm{Func}(\mathcal{E}G, \varphi\text{-frame})|$. On the other hand, G acts on $\mathcal{E}G$ on the right (by left-multiplication of the inverse), and, by precomposition, this induces a left-action of G on $|\,\mathrm{Func}(\mathcal{E}G, \varphi\text{-frame})|$. The respective actions of G and $\mathrm{GL}_n(F)$ on $|\,\mathrm{Func}(\mathcal{E}G, \varphi\text{-frame})|$ are clearly compatible. It follows that, in order to prove that $|\,\mathrm{Func}(\mathcal{E}G, \varphi\text{-frame})| \longrightarrow |\,\mathrm{Func}(\mathcal{E}G, \varphi\text{-mod})|$ is a $(G, \mathrm{GL}_n(F))$-principal bundle, it suffices to produce local trivializations of it.

The proof is split into four parts. In the first one, we construct "structural" maps that are linked to the topological categories of Section 2, and we prove that they are continuous. This will be used to prove that the local trivialization maps that will be constructed later are actually continuous. In Step 2, we construct an open covering of the space $|\,\mathrm{Func}(\mathcal{E}G, \varphi\text{-mod})|$, and use it in Step 3 to produce local sections, and then local trivializations. All the results from Theorem 2.6 are finally deduced in Step 4. Before moving on to Step 1, we will need to recall some notations on simplicial spaces and prove a basic lemma on fibre bundles.

A.1 Basic notation

For $n \in \mathbb{N}$, we let $\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_0 + \cdots + t_n = 1\}$ denote the standard n-simplex, and $\partial \Delta^n$ its boundary.

We let Δ denote the simplicial category. For $N \in \mathbb{N}$ and $i \in [N+1]$, $\delta_i^N : [N] \hookrightarrow [N+1]$ will denote the face morphism whose image does not contain i,

whilst, for $N \in \mathbb{N}$ and $i \in [N]$, $\sigma_i^{N+1} : [N+1] \twoheadrightarrow [N]$ the degeneracy morphism such that $\sigma_i^N(i) = \sigma_i^N(i+1)$. If $A = (A_n)_{n\in\mathbb{N}}$ is a simplicial space, $N \in \mathbb{N}$ and $i \in [N+1]$, the face map associated to δ_i^N is denoted by $d_i^N : A_{N+1} \to A_N$ and, for $N \in \mathbb{N}^*$ and $i \in [N-1]$, the degeneracy map associated to σ_i^N is denoted by $s_i^N : A_{N-1} \to A_N$. When $A = (A_n)_{n\in\mathbb{N}}$ is a simplicial space, $B = (B_n)_{n\in\mathbb{N}}$ is a cosimplicial space, and $f \in \mathrm{Hom}(\boldsymbol{\Delta})$, we let f_* (respectively f^*) denote the map associated to f in A (resp. in B). When A is a simplicial space, recall that its thin geometric realization is the quotient space

$$|A| := \left(\coprod_{n\in\mathbb{N}} (A_n \times \Delta^n) \right) / \sim$$

where $\sim$ is defined by: $(x, f^*(y)) \sim (f_*(x), y)$ for all $f : [n] \to [m]$, all $x \in A_m$ and all $y \in \Delta^n$.

A.2 A fundamental lemma

Lemma A.2. *Let X be a locally-countable CW-complex, and $\tilde{\varphi} : \tilde{E} \to X$ be a $GL_n(F)$-principal bundle (with corresponding vector bundle φ). Then:*

*(i) There is a continuous map $\delta : \mathrm{Hom}(\varphi\text{-mod}) = (\tilde{E} \times \tilde{E})/GL_n(F) \to \mathbb{R}_+$, called a **separation map** such that*

$$\forall(x,y) \in \tilde{E} \times \tilde{E}, \delta([x,y]) = 0 \Leftrightarrow x = y. \tag{2}$$

(ii) For every $x \in X$, there is a continuous section $s_x : X \longrightarrow E^{\oplus n}$ such that $s_x(x) \in \tilde{E}$.

Remark 16. Property (2) means that δ vanishes precisely on the set of identity morphisms of the category φ-mod.

Proof. Notice first that given a relative CW-complex (Y, B), any continuous map $\alpha : B \to \mathbb{R}_+$ may be extended to a continuous map $\tilde{\alpha} : Y \to \mathbb{R}_+$ so that $\tilde{\alpha}^{-1}\{0\} = \alpha^{-1}\{0\}$: this is easily done by working cell by cell. We will use this remark to construct a continuous map $\alpha : X \times X \to \mathbb{R}_+$ such that

$$\alpha^{-1}\{0\} = \Delta_X := \big\{ (x,x) \mid x \in X \big\}.$$

Notice that $X \times X$ is a CW-complex for the cartesian product topology (since X is locally-countable). Let $n \in \mathbb{N}$ and assume that α has been defined on the $(n-1)$-th skeleton of $X \times X$. Let $\Delta_X^{(n)}$ denote the intersection of Δ_X with the n-th skeleton $\mathrm{Sk}_n(X \times X)$ of $X \times X$. It is however easily shown that $\big(\mathrm{Sk}_n(X \times X), \Delta_X^{(n)} \cup \mathrm{Sk}_{n-1}(X \times X)\big)$ is a relative CW-complex, using the somewhat obvious fact that $(\Delta^n \times \Delta^n, \Delta_{\Delta^n} \cup \partial(\Delta^n \times \Delta^n))$ is a finite relative CW-complex for every $n \in \mathbb{N}$.

We then extend α, first on $\Delta_X^{(n)} \cup \mathrm{Sk}_{n-1}(X \times X)$ by mapping any $x \in \Delta_X^{(n)}$ to 0, and then on $\mathrm{Sk}_n(X \times X)$ using the initial remark. Then $\forall x \in \mathrm{Sk}_n(X \times X)$, $\alpha(x) =$

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$0 \Leftrightarrow x \in \Delta_X$. This induction process then defines α on the whole $X \times X$ with the claimed property.

We now choose a continuous map $\alpha : X \times X \to \mathbb{R}_+$ such that $\alpha^{-1}\{0\} = \Delta_X$. We also choose a system $(U_i, \varphi_i)_{i \in I}$ of local trivializations for the principal bundle $\tilde{\varphi}$ (where $\varphi_i : \tilde{E}_{|U_i} \overset{\cong}{\Rightarrow} U_i \times \mathrm{GL}_n(F)$, by convention) together with a partition of unity $(\alpha_i)_{i \in I}$ for the covering $(U_i)_{i \in I}$ of X. For any $i \in I$, we will let $\pi_2 : U_i \times \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ denote the projection onto the second factor. Finally, we choose a norm N on the linear space $\mathrm{M}_n(F)$ of square matrices.

For $(x, y) \in (\mathrm{GL}_n(F))^2$, we set

$$\delta_1(x, y) := N(xy^{-1} - I_n)$$

and notice that δ_1 is continuous and invariant by the right-action of $\mathrm{GL}_n(F)$ on itself. Notice also that

$$\forall (x, y) \in \mathrm{GL}_n(F)^2, \ \delta_1(x, y) \geq 0 \quad \text{and} \quad \delta_1(x, y) = 0 \Leftrightarrow x = y.$$

For $(x, y) \in \tilde{E} \times \tilde{E}$, we then set:

$$\delta(x, y) := \alpha\big(\tilde{\varphi}(x), \tilde{\varphi}(y)\big) + \sum_{i \in I} \alpha_i\,(\tilde{\varphi}(x))\,\alpha_i\,(\tilde{\varphi}(y)) \inf\Big[1, \delta_1\big((\pi_2 \circ \varphi_i)(x), (\pi_2 \circ \varphi_i)(y)\big)\Big].$$

From there, it is straightforward to prove that $\delta : \tilde{E} \times \tilde{E} \to \mathbb{R}_+$ is continuous, that $\delta^{-1}\{0\} = \Delta_{\tilde{E}}$ and that $\forall (M, x, y) \in \mathrm{GL}_n(F) \times \tilde{E}^2, \ \delta(x.M, y.M) = \delta(x, y)$, which shows that δ induces a continuous map $(\tilde{E} \times \tilde{E})/\mathrm{GL}_n(F) \to \mathbb{R}_+$ which satisfies condition (2). This proves statement (i).

We now choose a system of local trivializations $(U_i, \psi_i)_{i \in I}$ for $\varphi : E \to X$, together with a partition of unity $(\alpha_i)_{i \in I}$ for covering $(U_i)_{i \in I}$ of X. We choose an $i_0 \in I$ such that $\alpha_{i_0}(x) > 0$. If $\theta_{i_0} : U_{i_0} \to E^{\oplus n}$ is the map which assigns $(\varphi_{i_0}(y, e_1), \ldots, \varphi_{i_0}(y, e_n))$ to y (where $(e_1, \ldots, e_n)$ denotes the canonical basis of F^n), then $s_x := \alpha_{i_0}.\theta_{i_0}$ has the required property for statement (ii). $\qquad\square$

A.3   Proof of Theorem 2.6

We fix a system $(U_i, \varphi_i)_{i \in I}$ of local trivializations of $\varphi : E \to X$.

A.3.1   Step 1: Structural maps

To make things easier, we rename

$$\mathcal{E} := \mathrm{Func}(\mathcal{E}G, \varphi\text{-mod}), \quad \mathcal{F} := \mathrm{Func}(\mathcal{E}G, \varphi\text{-frame}) \quad \text{and} \quad \mathcal{G} := \mathrm{Func}(\mathcal{E}G, \varphi\text{-Bdl}).$$

For every $m \in \mathbb{N}$ and $g \in G$, there is a canonical map

$$\alpha_{g,m} : \begin{cases} \mathcal{N}(\mathcal{E})_m & \longrightarrow X \\ F_0 \to \cdots \to F_m & \longmapsto F_0(g), \end{cases}$$

and we let $\mathcal{N}(\mathcal{E})_m \underset{g,0,X}{\times} \tilde{E}$ denote the limit of the diagram $\mathcal{N}(\mathcal{E})_m \overset{\alpha_{g,m}}{\longrightarrow} X \overset{\tilde{\varphi}}{\longleftarrow} \tilde{E}$.

We then let

$$\psi_{g,m} : \mathcal{N}(\mathcal{E})_m \underset{g,0,X}{\times} \tilde{E} \longrightarrow \mathcal{N}(\mathcal{F})_m$$

denote the map which assigns $F'_0 \to \cdots \to F'_m$ to the compatible pair $(F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m, \mathbf{B})$, where $F'_k(h) = [\eta_{k-1} \circ \cdots \circ \eta_0] \circ F_0((g,h))[\mathbf{B}]$ for $k \in \{0, \ldots, m\}$ and $h \in G$.

One should think of an element $F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m$ of $\mathcal{N}(\mathcal{E})$ as an array of fibers of φ, with an isomorphism between any pair of fibers in the array, such that the whole diagram is commutative. On the other hand, an element $F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m$ of $\mathcal{N}(\mathcal{F})$ should simply be thought of as an array of basis of fibers of φ. Let $F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m$ in $\mathcal{N}(\mathcal{E})$, a basis $\mathbf{B}$ of the fiber $E_{g,0}$, and say we wish to obtain an element of $\mathcal{N}(\mathcal{F})$. If we plug the basis $\mathbf{B}$ at the position $(g,0)$ in the diagram associated to $F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m$, then we can use the isomorphisms of the diagram to recover a basis in every position (h,i) of the diagram. The remaining diagram of basis corresponds to an element of $\mathcal{N}(\mathcal{F})$, and this is precisely the image of the pair $(F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m, \mathbf{B})$ by $\psi_{g,m}$, judging from the previous definition.

Lemma A.3. *Let $f : [N] \to [N']$ be a morphism in Δ, let $g \in G$, $x = F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{N'-1}}{\to} F_{N'} \in \mathcal{N}(\mathcal{E})_{N'}$, and $y \in \tilde{E}$ such that $\tilde{\varphi}(y) = \alpha_{g,N'}(x)$. Then*

$$f_*\big(\psi_{g,N'}(x,y)\big) = \psi_{g,N}\big(f_*(x), (\eta_{f(0)-1}(g) \circ \cdots \circ \eta_0(g))(y)\big)$$

with the convention that $\eta_{f(0)-1} \circ \cdots \circ \eta_0 = \mathrm{id}_{E_{F_0(g)}}$ whenever $f(0) = 0$.

Proof. By definition, $\psi_{g,N}(f_*(x), y)$ is the only N-simplex of $\mathcal{N}(\mathcal{F})$ which is sent to $f_*(x)$ by $\mathcal{N}(\mathcal{F}) \to \mathcal{N}(\mathcal{E})$ and such that the basis in position $(g,0)$ is $(\eta_{f(0)-1}(g) \circ \cdots \circ \eta_0(g))(y)$. On the other hand, $\psi_{g,N'}(x,y)$ is the only N'-simplex of $\mathcal{N}(\mathcal{F})$ which is sent to x by $\mathcal{N}(\mathcal{F}) \to \mathcal{N}(\mathcal{E})$ and such that the basis in position $(g,0)$ is y. Hence $f_*(\psi_{g,N'}(x,y))$ is the only N-simplex of $\mathcal{N}(\mathcal{F})$ which is sent to $f_*(x)$ by $\mathcal{N}(\mathcal{F}) \to \mathcal{N}(\mathcal{E})$ and such that the basis in position $(g,0)$ is $(\eta_{f(0)-1}(g) \circ \cdots \circ \eta_0(g))(y)$. $\square$

We have another map

$$\chi_m : \mathcal{N}(\mathcal{F})_m \underset{\mathcal{N}(\mathcal{E})_m}{\times} \mathcal{N}(\mathcal{F})_m \longrightarrow \mathrm{GL}_n(F),$$

which sends a (compatible) pair $\big((F_0 \to \cdots \to F_m), (F'_0 \to \cdots \to F'_m)\big)$ to the unique $M \in \mathrm{GL}_n(F)$ such that $F_k(g) = F'_k(g).M$ for every $k \in \{0, \ldots, m\}$ and $g \in G$. We also define

$$v_m : \mathcal{N}(\mathcal{F})_m \times F^n \longrightarrow \mathcal{N}(\mathcal{G})_m,$$

which maps a pair $(F_0 \to \cdots \to F_m, x)$ to $F'_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F'_m$, where:

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(i) $F_k'(g) = \theta(F_k(g), x)$ for all $(k, g) \in \{0, \dots, m\} \times G$;

(ii) $F_k'(g, g') = (F_k'(g), F_k'(g'), F_k(g) \mapsto F_k(g'))$, for all $(k, g, g') \in \{0, \dots, m\} \times G^2$;

(iii) $\eta_k(g) = (F_k'(g), F_{k+1}'(g), F_k(g) \mapsto F_{k+1}(g))$ for all $(k, g) \in \{0, \dots, m-1\} \times G$.

Finally, we define

$$\varepsilon_m : \mathcal{N}(\mathcal{G})_m \underset{\mathcal{N}(\mathcal{E})_m}{\times} \mathcal{N}(\mathcal{F})_m \longrightarrow F^n$$

as the map which sends every (compatible) pair $((F_0 \to \cdots \to F_n), (F_0' \to \cdots \to F_m'))$ to the unique $x \in F^n$ such that $F_0'(1_G) = \theta(F_0(1_G), x)$, i.e. the unique $x \in F^n$ such that $\nu_m((F_0' \to \cdots \to F_m'), x) = F_0 \to \cdots \to F_m$.

Proposition A.4. *For every $m \in \mathbb{N}$ and $g \in G$, the maps $\psi_{g,m}$, χ_m, v_m and ε_m are continuous.*

Proof. All products and spaces of maps will be formed in the category of k-spaces.
• **Continuity of χ_m.** The map χ_m may be decomposed as

$$\mathcal{N}(\mathcal{F})_m \underset{\mathcal{N}(\mathcal{E})_m}{\times} \mathcal{N}(\mathcal{F})_m \xrightarrow{(1_G,0)\times(1_G,0)} \tilde{E} \underset{X}{\times} \tilde{E} \longrightarrow \mathrm{GL}_n(F),$$

where the map $(1_G, 0)$ assigns $F_0(1_G)$ to $F_0 \to \cdots \to F_m$. It follows that the first map is continuous, and the second map also is since its restrictions over the open sets U_i are continuous.
• **Continuity of v_m.** It obviously suffices to prove that v_0 is continuous. However, the canonical map $\tilde{E} \times F^n \to E$ is continuous, hence the continuity of

$$\begin{cases} (\tilde{E} \times \tilde{E})^{G\times G} \times F^n & \longrightarrow (\mathrm{Hom}(\varphi\text{-Bdl}))^{G\times G} \\ (f_1, f_2, x) & \longmapsto \big[(g_1, g_2) \mapsto (f_1(g_1, g_2).x, f_2(g_1, g_2).x, [f_1(g_1, g_2) \mapsto f_2(g_1, g_2)])\big]. \end{cases}$$

Therefore, v_0, being the restriction of this map to $\mathcal{N}(\mathcal{F})_0 \times F^n$, is continuous.
• **Continuity of ε_m.** The map ε_m may be seen as the composite:

$$\mathcal{N}(\mathcal{G})_m \underset{\mathcal{N}(\mathcal{E})_m}{\times} \mathcal{N}(\mathcal{F})_m \xrightarrow{(1_G,0)\times(1_G,0)} E \underset{X}{\times} \tilde{E} \longrightarrow F^n,$$

where the second map sends any compatible pair $(x, \mathbf{B})$ to the unique $y \in F^n$ such that $\theta(\mathbf{B}, y) = x$. The continuity of the first map is proven in the same way as the continuity of χ_m. The second map is continuous because its restrictions over the open subsets U_i of X clearly are. Therefore ε_m is continuous.
• **Continuity of $\psi_{g,m}$.** Let $\mathrm{Hom}(\varphi\text{-mod}) \underset{X}{\times} \tilde{E}$ denote the subspace of the product space $\mathrm{Hom}(\varphi\text{-mod}) \times \tilde{E}$ consisting of the pairs (f, y) such that $\mathrm{In}_{\varphi\text{-mod}}(f) = \tilde{\varphi}(y)$, and let $(\tilde{E} \times \tilde{E}) \underset{X}{\times} \tilde{E}$ denote the subspace of $\tilde{E} \times \tilde{E} \times \tilde{E}$ consisting of the triples (x, y, z) such that $\tilde{\varphi}(x) = \tilde{\varphi}(z)$.

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We start by proving that the following map is continuous:

$$\begin{cases} \mathrm{Hom}(\varphi\text{-mod}) \underset{X}{\times} \tilde{E} & \longrightarrow \tilde{E} \\ (f, \mathbf{B}) & \longmapsto f(\mathbf{B}). \end{cases}$$

Notice first that the projection $(\tilde{E} \times \tilde{E}) \underset{X}{\times} \tilde{E} \longrightarrow \mathrm{Hom}(\varphi\text{-mod}) \underset{X}{\times} \tilde{E}$ is an open identification map. It then suffices to prove that the map

$$\begin{cases} (\tilde{E} \times \tilde{E}) \underset{X}{\times} \tilde{E} & \longrightarrow \tilde{E} \\ (\mathbf{B}.M, \mathbf{B}', \mathbf{B}), M \in \mathrm{GL}_n(F) & \longmapsto \mathbf{B}'.M^{-1} \end{cases}$$

is continuous, which is a straightforward task using local trivializations of $\tilde{\varphi}$.

We may now prove that $\psi_{g,0}$ is continuous. We denote by $(\mathrm{Hom}(\varphi\text{-mod}))^G \underset{G,X}{\times} \tilde{E}$ (resp. by $(\mathrm{Hom}(\varphi\text{-mod}))^{G \times G} \underset{G,X}{\times} \tilde{E}$) the subset of $(\mathrm{Hom}(\varphi\text{-mod}))^G \times \tilde{E}$ consisting of the pairs (f, y) such that $\forall g' \in G$, $\mathrm{In}(f(g')) = \tilde{\varphi}(y)$ (resp. $\forall g' \in G$, $\mathrm{In}(f(g, g')) = \tilde{\varphi}(y)$). We consider then the composite map

$$(\mathrm{Hom}(\varphi\text{-mod}))^{G \times G} \underset{G,X}{\times} \tilde{E} \longrightarrow (\mathrm{Hom}(\varphi\text{-mod}))^G \underset{G,X}{\times} \tilde{E} \longrightarrow \tilde{E}^G \longrightarrow \tilde{E}^G \times \tilde{E}^G \longrightarrow (\tilde{E} \times \tilde{E})^{G \times G},$$

where the first map is obtained by precomposition with the injection $\begin{cases} G & \to G \times G \\ g_1 & \mapsto (g, g_1), \end{cases}$ and is thus continuous. The last map is continuous due to classic results on k-spaces, whilst the third one is simply the diagonal map. It remains to prove that the map

$$\left(\mathrm{Hom}(\varphi\text{-mod})\right)^G \underset{G,X}{\times} \tilde{E} \longrightarrow \tilde{E}^G$$

is continuous. However, it may be regarded upon as a composite

$$(\mathrm{Hom}(\varphi\text{-mod}))^G \underset{G,X}{\times} \tilde{E} \longrightarrow (\mathrm{Hom}(\varphi\text{-mod}))^G \underset{G,X}{\times} \tilde{E}^G \longrightarrow \left(\mathrm{Hom}(\varphi\text{-mod}) \underset{X}{\times} \tilde{E}\right)^G \longrightarrow \tilde{E}^G.$$

The first map is continuous, since it comes from $\begin{cases} \tilde{E} & \to \tilde{E}^G \\ y & \mapsto [g \mapsto y], \end{cases}$ the second one is also continuous by the usual properties of k-spaces, and we have already proven that the third one is continuous. Hence $\psi_{g,0}$ is continuous.

Finally, in order to prove the continuity of $\psi_{g,m}$ for an arbitrary $m \in \mathbb{N}$, it suffices to prove the continuity of the map obtained by composing $\psi_{g,m}$ with the inclusion of $\mathcal{N}(\mathcal{F})_m$ into $(\tilde{E})^{\{0,\dots,m\} \times G}$. This is done is the same way as in the case $m = 0$. $\qquad \square$

The maps χ_m, for $m \in \mathbb{N}$, induce a morphism of simplicial spaces $\mathcal{N}(\mathcal{F}) \underset{\mathcal{N}(\mathcal{E})}{\times} \mathcal{N}(\mathcal{F}) \longrightarrow \mathrm{GL}_n(F)$, hence a continuous map:

$$\chi : |\mathcal{F}| \underset{|\mathcal{E}|}{\times} |\mathcal{F}| \longrightarrow \mathrm{GL}_n(F).$$

In the same manner, the maps ε_m, for $m \in \mathbb{N}$, induce a morphism of simplicial spaces $\mathcal{N}(\mathcal{G}) \underset{\mathcal{N}(\mathcal{E})}{\times} \mathcal{N}(\mathcal{F}) \longrightarrow F^n$, hence a continuous map:

$$\varepsilon : |\mathcal{G}| \underset{|\mathcal{E}|}{\times} |\mathcal{F}| \longrightarrow F^n.$$

Finally, the maps v_m, for $m \in \mathbb{N}$, induce a morphism of simplicial spaces $\mathcal{N}(\mathcal{F}) \times F^n \longrightarrow \mathcal{N}(\mathcal{G})$, hence a continuous map

$$v : |\mathcal{F}| \times F^n \longrightarrow |\mathcal{G}|.$$

A.3.2   Step 2: An open cover of $|\operatorname{Func}(\mathcal{E}G, \varphi\text{-mod})|$

By Lemma A.2, we may choose $\delta : \operatorname{Hom}(\varphi\text{-mod}) \to \mathbb{R}_+$ satisfying (2) and, for every $x_1 \in X$, a section s_{x_1} which satisfies the requirements of statement (ii).

Let $m \in \mathbb{N}$ and $(g_0, g_1, \ldots, g_{m-1}) \in G^m$. Then, for every $0 \leq i_0 < \cdots < i_m \leq N$, we define the map

$$\alpha^{(i_0,\ldots,i_m)}_{N,x_1,(g_0,\ldots,g_{m-1})} : \mathcal{N}(\mathcal{E})_N \times \Delta^N \longrightarrow E^{\oplus n}$$

as the one which maps any pair $\left((F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{N-1}}{\to} F_N), (t_0, \ldots, t_N) \right)$ to

$$t_{i_0} \left[\prod_{j=1}^{m} t_{i_j} \delta \left[\eta_{i_{(j-1)}}(g_{j-1}) \circ \cdots \circ \eta_{i_j - 1}(g_{j-1}) \right] \right] (\eta_{i_0 - 1}(g_0) \circ \cdots \circ \eta_0(g_0))^{-1} (s_{x_1}(F_{i_0}(g_0))).$$

We then set:

$$\alpha_{N,x_1,(g_0,\ldots,g_{m-1})} : \begin{cases} \mathcal{N}(\mathcal{E})_N \times \Delta^N & \longrightarrow E^{\oplus n} \\ (x,t) & \longmapsto \displaystyle\sum_{0 \leq i_0 < \cdots < i_m \leq N} \alpha^{(i_0,\ldots,i_m)}_{N,x_1,(g_0,\ldots,g_{m-1})}(x,t). \end{cases}$$

Notice that $\alpha_{N,x_1,(g_0,\ldots,g_{m-1})}$ is continuous and, for any $x = F_0 \to \cdots \to F_N$ and any $t \in \Delta^N$, one has $\alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(x,t) \in (E_{F_0(g_0)})^n$.

Set now
$$U^{(N)}_{x_1,(g_0,\ldots,g_{m-1})} := (\alpha_{N,x_1,(g_0,\ldots,g_{m-1})})^{-1}(\tilde{E})$$

and notice that this is an open subset of $\mathcal{N}(\mathcal{E})_N \times \Delta^N$ since $\tilde{E}$ is an open subset of $E^{\oplus n}$. Denote by $\pi_N : \mathcal{N}(\mathcal{E})_N \times \Delta^N \to |\mathcal{E}|$ the canonical projection.

Proposition A.5.

(i) $U_{x_1,(g_0,\ldots,g_{m-1})} := \displaystyle\bigcup_{N=0}^{\infty} \pi_N(U^{(N)}_{x_1,(g_0,\ldots,g_{m-1})})$ is an open subset of $|\mathcal{E}|$.

(ii) $\forall N \in \mathbb{N}$, $\pi_N^{-1}(U_{x_1,(g_0,\ldots,g_{m-1})}) = U^{(N)}_{x_1,(g_0,\ldots,g_{m-1})}$.

(iii) $\forall (N, N') \in \mathbb{N}^2, \forall f \in \mathrm{Hom}_\Delta([N], [N']), \forall (x = F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{N'-1}}{\to} F_{N'}, t) \in \mathcal{N}(\mathcal{E})_{N'} \times \Delta^N$,

$$\alpha_{N, x_1, (g_0, \ldots, g_{m-1})}(f_*(x), t) = (\eta_{f(0)-1}(g_0) \circ \cdots \circ \eta_0(g_0)) \left(\alpha_{N', x_1, (g_0, \ldots, g_{m-1})}(x, f^*(t)) \right)$$

with the convention that $\eta_{f(0)-1}(g_0) \circ \cdots \circ \eta_0(g_0) = \mathrm{id}_{E_{F_0(g_0)}}$ when $f(0) = 0$.

(iv) $(U_{x_1, (g_0, \ldots, g_{m-1})})_{x_1 \in X, m \in \mathbb{N}, (g_0, \ldots, g_{m-1}) \in G^m}$ is a cover of $|\mathcal{E}|$.

Proof. We first prove (ii). It suffices to prove the following facts:

$$\forall N \in \mathbb{N}, \forall (x, t) \in \mathcal{N}(\mathcal{E})_N \times \Delta^{N+1}, \forall i \in [N],$$
$$(s_i^{N+1}(x), t) \in U_{x_1, (g_0, \ldots, g_{m-1})}^{(N+1)} \Leftrightarrow (x, \sigma_i^{N+1}(t)) \in U_{x_1, (g_0, \ldots, g_{m-1})}^{(N)} \quad (3)$$

and

$$\forall N \in \mathbb{N}, \forall (x, t) \in \mathcal{N}(\mathcal{E})_{N+1} \times \Delta^N, \forall i \in [N+1],$$
$$(x, \delta_i^N(t)) \in U_{x_1, (g_0, \ldots, g_{m-1})}^{(N+1)} \Leftrightarrow (d_i^N(x), t) \in U_{x_1, (g_0, \ldots, g_{m-1})}^{(N)}. \quad (4)$$

Let $0 \le i_0 < \cdots < i_m \le N+1$, let $i \in [N]$, $x \in \mathcal{N}(\mathcal{E})_N$ and $t \in \Delta^{N+1}$. If there exists an index j such that $i_j = i$ and $i_{j+1} = i+1$, then $\delta(\eta_j) = \delta(\mathrm{id}_{E_{F_{i_j}(g_j)}}) = 0$, and so

$$\alpha_{N+1, x_1, (g_0, \ldots, g_{m-1})}^{(i_0, \ldots, i_m)}(s_i^{N+1}(x), t) = 0.$$

If there exists an index j such that $i_j = i$ and $i_{j+1} > i+1$, we set $i_l' = i_l$ when $l \ne j$, and $i_j' = i+1$. Also, for $l \in [m]$, we set $i_l'' = \sigma_i^{N+1}(i_l)$. We may then check that

$$\alpha_{N+1, x_1, (g_0, \ldots, g_{m-1})}^{(i_0, \ldots, i_m)}(s_i^{N+1}(x), t) + \alpha_{N+1, x_1, (g_0, \ldots, g_{m-1})}^{(i_0', \ldots, i_m')}(s_i^{N+1}(x), t) = \alpha_{N, x_1, (g_0, \ldots, g_{m-1})}^{(i_0'', \ldots, i_m'')}(x, \sigma_i^{N+1}(t)).$$

If $\{i, i+1\} \cap \{i_j, 0 \le j \le m\} = \emptyset$, we still set $i_l'' = \sigma_i^{N+1}(i_l)$ for any $l \in [m]$. It follows that

$$\alpha_{N+1, x_1, (g_0, \ldots, g_{m-1})}^{(i_0, \ldots, i_m)}(s_i^{N+1}(x), t) = \alpha_{N, x_1, (g_0, \ldots, g_{m-1})}^{(i_0'', \ldots, i_m'')}(x, \sigma_i^{N+1}(t)).$$

Summing the previous equalities yields

$$\alpha_{N+1, x_1, (g_0, \ldots, g_{m-1})}(s_i^{N+1}(x), t) = \alpha_{N, x_1, (g_0, \ldots, g_{m-1})}(x, \sigma_i^{N+1}(t))$$

and (3) follows right away.

Let $0 \le i_0 < \cdots < i_m \le N+1$, $i \in [N+1]$, $x \in \mathcal{N}(\mathcal{E})_{N+1}$ and $t \in \Delta^N$. We write $x = F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_N}{\to} F_N$. If there exists an index j such that $i_j = i$, then

$$\alpha_{N+1, x_1, (g_0, \ldots, g_{m-1})}^{(i_0, \ldots, i_m)}(x, \delta_i^N(t)) = 0.$$

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Otherwise, we set $i'_j = \sigma_i^{N+1}$ for all $j \in [m]$.
When $i > 0$,

$$\alpha_{N+1,x_1,(g_0,\ldots,g_{m-1})}^{(i_0,\ldots,i_m)}(x, \delta_i^N(t)) = \alpha_{N,x_1,(g_0,\ldots,g_{m-1})}^{(i'_0,\ldots,i'_m)}(d_i^N(x), t).$$

Also

$$\eta_0(g_0)\left(\alpha_{N+1,x_1,(g_0,\ldots,g_{m-1})}^{(i_0,\ldots,i_m)}(x, \delta_0^N(t))\right) = \alpha_{N,x_1,(g_0,\ldots,g_{m-1})}^{(i'_0,\ldots,i'_m)}(d_0^N(x), t).$$

Summing these equalities when yields

$$\begin{cases} i > 0 \;\Rightarrow\; \alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(d_i^N(x), t) = \alpha_{N+1,x_1,(g_0,\ldots,g_{m-1})}(x, \delta_i^N(t)) \\ \alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(d_0^N(x), t) = \eta_0(g_0)\left(\alpha_{N+1,x_1,(g_0,\ldots,g_{m-1})}(x, \delta_0^N(t))\right). \end{cases}$$

and (4) follows right away. This proves both statements (ii) and (iii), and (i) then follows from (ii) and the fact that $U_{x_1,(g_0,\ldots,g_{m-1})}^{(N)}$ is an open subset of $\mathcal{N}(\mathcal{E})_N \times \Delta^N$.

Let $y \in |\mathcal{E}|$. Then there exists an integer $m \in \mathbb{N}$, a non-degenerate m-simplex $x \in \mathcal{N}(\mathcal{E})_m$ and a point $\alpha \in \Delta^m \smallsetminus \partial\Delta^m$ such that $y = \pi_m(x, \alpha)$. Since $x = F_0 \overset{\eta_0}{\to} \ldots \overset{\eta_{m-1}}{\to} F_m$ is non-degenerate, there exists, for all $i \in [m-1]$, an element $g_i \in G$ such that $\delta(\eta_i(g_i)) \neq 0$. We simply remark that $(x, \alpha) \in U_{F_0(g_0),(g_0,\ldots,g_{m-1})}$, which proves statement (iv). $\qquad\square$

The open subsets $U_{x_1,(g_0,\ldots,g_{m-1})}$ are defined in such a way that, for every $N \in \mathbb{N}$, and every $(x, t) \in \mathcal{N}(\mathcal{E})_N \times \Delta^N$ such that $[x, t] \in U_{x_1,(g_0,\ldots,g_{m-1})}$: if we write $x = F_0 \to \cdots \to F_N$, then the map $\alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(x, t)$ provides a continuous way (with respect to (x, t)) to choose a basis in the fiber of $F_0(g_0)$. In the next step, this is used, in conjunction with the maps $\psi_{g,m}$, to construct local sections of the bundle $|\mathcal{F}| \to |\mathcal{E}|$.

A.3.3 Step 3: Local trivializations

We let $\pi : |\mathcal{F}| \to |\mathcal{E}|$ denote the map induced by the functor $\text{Func}(\mathcal{E}G, \varphi\text{-frame}) \longrightarrow \text{Func}(\mathcal{E}G, \varphi\text{-mod})$ discussed earlier. We will now write

$$\alpha_{N,x_1,(g_0,\ldots,g_{m-1})} : U_{x_1,(g_0,\ldots,g_{m-1})}^{(N)} \longrightarrow \tilde{E}$$

when we actually mean the restriction of $\alpha_{N,x_1,(g_0,\ldots,g_{m-1})}$ to $U_{x_1,(g_0,\ldots,g_{m-1})}^{(N)}$.

We then consider, for every $N \in \mathbb{N}$, the composite map

$$\beta_{x_1,(g_0,\ldots,g_{m-1})}^{(N)} : U_{x_1,(g_0,\ldots,g_{m-1})}^{(N)} \longrightarrow \left[\mathcal{N}(\mathcal{E})_N \underset{g_0,0,X}{\times} \tilde{E}\right] \times \Delta^N \xrightarrow{\psi_{g_0,N} \times \text{id}_{\Delta^N}} \mathcal{N}(\mathcal{F})_N \times \Delta^N,$$

where the first map assigns $(x, \alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(x, t), t)$ to (x, t). Proposition A.4 shows $\beta_{x_1,(g_0,\ldots,g_{m-1})}^{(N)}$ is continuous.

Proposition A.6. *Let $x_1 \in X$, $m \in \mathbb{N}$, and $(g_0, \ldots, g_{m-1}) \in G^m$.*

(i) The maps $\beta^{(N)}_{x_1,(g_0,\ldots,g_{m-1})}$, for $N \in \mathbb{N}$, induce a continuous map

$$\beta_{x_1,(g_0,\ldots,g_{m-1})} : U_{x_1,(g_0,\ldots,g_{m-1})} \longrightarrow |\mathcal{F}|.$$

(ii) The map $\beta_{x_1,(g_0,\ldots,g_{m-1})}$ is a local section of π.

(iii) The map

$$\varphi_{x_1,(g_0,\ldots,g_{m-1})} : \begin{cases} U_{x_1,(g_0,\ldots,g_{m-1})} \times GL_n(F) & \longrightarrow \pi^{-1}(U_{x_1,(g_0,\ldots,g_{m-1})}) \\ (x, M) & \longmapsto \beta_{x_1,(g_0,\ldots,g_{m-1})}(x).M \end{cases}$$

is a homeomorphism over $U_{x_1,(g_0,\ldots,g_{m-1})}$.

Proof. (i) Let $(N, N') \in \mathbb{N}^2$, $f \in \mathrm{Hom}_\Delta([N],[N'])$, and $(x,t) \in \mathcal{N}(\mathcal{E})_{N'} \times \Delta^N$ such that $(x, f^*(t)) \in U^{(N')}_{x_1,(g_0,\ldots,g_{m-1})}$. If $\beta^{(N')}_{x_1,(g_0,\ldots,g_{m-1})}(x, f^*(t)) = (y, s)$ and $\beta^{(N)}_{x_1,(g_0,\ldots,g_{m-1})}(f_*(x), t) = (y', s')$, we have to prove that $s = f^*(s')$ and $y' = f_*(y)$. The first identity is obvious from the definition of $\beta^{(N)}_{x_1,(g_0,\ldots,g_{m-1})}$. The second one may be restated as:

$$f_*\left(\psi_{g_0,N'}(x, \alpha_{N',x_1,(g_0,\ldots,g_{m-1})}(x, f^*(t)))\right) = \psi_{g_0,N}(f_*(x), \alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(f_*(x), t))$$
$$(5)$$

However $\alpha_{N,x_1,(g_0,\ldots,g_{m-1})}(f_*(x), t) = (\eta_{f(0)-1}(g_0) \circ \cdots \circ \eta_0(g_0))\left(\alpha_{N',x_1,(g_0,\ldots,g_{m-1})}(x, f^*(t))\right)$. hence (5) follows from Lemma A.3. This proves (i) since the maps $\beta^{(N)}_{x_1,(g_0,\ldots,g_{m-1})}$ are continuous.

(ii) is an obvious consequence of the definition of $\beta_{x_1,(g_0,\ldots,g_{m-1})}$.

(iii) The map $\varphi_{x_1,(g_0,\ldots,g_{m-1})}$ is clearly continuous, and clearly bijective since $\beta_{x_1,(g_0,\ldots,g_{m-1})}$ is a local section of π. It remains to prove the continuity of its inverse map: the composite of it with the projection on the first factor is the continuous map $\pi_{|\pi^{-1}(U_{x_1,(g_0,\ldots,g_{m-1})})}$. The composite with the projection on the second factor is the map

$$\begin{cases} \pi^{-1}(U_{x_1,(g_0,\ldots,g_{m-1})}) & \longrightarrow GL_n(F) \\ x & \longmapsto M \quad \text{such that} \quad x = \beta_{x_1,(g_0,\ldots,g_{m-1})}(x).M \end{cases}$$

Setting $V := U_{x_1,(g_0,\ldots,g_{m-1})}$, it may be seen as the composite map

$$\pi^{-1}(V) \xrightarrow{(\mathrm{id}, \beta_{x_1,(g_0,\ldots,g_{m-1})} \circ \pi)} \pi^{-1}(V) \underset{V}{\times} \pi^{-1}(V) \xrightarrow{\chi} GL_n(F),$$

and is thus continuous. Therefore $\varphi_{x_1,(g_0,\ldots,g_{m-1})}$ is a homeomorphism. $\square$

We conclude that $\widetilde{\mathrm{Vec}}^\varphi_G \to \mathrm{Vec}^\varphi_G$ is a $(G, GL_n(F))$-principal bundle.

A.3.4   Step 4: $E\operatorname{Vec}_G^{\varphi} \to \operatorname{Vec}_G^{\varphi}$ as a G-vector bundle

The fibers of $\pi' : E\operatorname{Vec}_G^{\varphi} \to \operatorname{Vec}_G^{\varphi}$ have natural structures of vector spaces which are inherited from those of the fibers of $E \to X$. The group G acts on the left on $E\operatorname{Vec}_G^{\varphi}$ and the projection $E\operatorname{Vec}_G^{\varphi} \to \operatorname{Vec}_G^{\varphi}$ is a G-map. Also, the action of G on $E\operatorname{Vec}_G^{\varphi}$ restricts to linear isomorphisms on the fibers.

Take arbitrary $x_1 \in X$, $m \in \mathbb{N}$ and $(g_0, \ldots, g_{m-1}) \in G^m$, and set $V := U_{x_1,(g_0,\ldots,g_{m-1})}$. We may then consider the composite map

$$\varphi'_{x_1,(g_0,\ldots,g_{m-1})} : V \times F^n \xrightarrow{\beta_{x_1,(g_0,\ldots,g_{m-1})} \times \operatorname{id}_{F^n}} \pi^{-1}(V) \times F^n \xrightarrow{v} (\pi')^{-1}(V).$$

Proposition A.7. *The map $\varphi'_{x_1,(g_0,\ldots,g_{m-1})}$ is a homeomorphism over V.*

Proof. The continuity, the bijectivity, and the fiberwise linearity of $\varphi'_{x_1,(g_0,\ldots,g_{m-1})}$ are clear. It remains to prove that the projection of the inverse map on the second factor is continuous. However this projection is no other than the (obviously continuous) composite map

$$(\pi')^{-1}(V) \xrightarrow{(\operatorname{id},\beta_{x_1,(g_0,\ldots,g_{m-1})})} (\pi')^{-1}(V) \underset{V}{\times} \pi^{-1}(V) \xrightarrow{\varepsilon} F^n.$$

$\square$

Obviously, the transitions between the trivialization maps constructed for π are identical to the transitions between the trivialization maps constructed for π'. Therefore, $E\operatorname{Vec}_G^{\varphi} \to \operatorname{Vec}_G^{\varphi}$ is an n-dimensional vector bundle, hence a G-vector bundle.

It remains to check that $\widetilde{\operatorname{Vec}}_G^{\varphi} \times_{\operatorname{GL}_n(F)} F^n \longrightarrow E\operatorname{Vec}_G^{\varphi}$ is an isomorphism of vector bundles over $\operatorname{Vec}_G^{\varphi}$ (since we already know that it is continuous and equivariant). This comes from the surjectivity of the map $\widetilde{\operatorname{Vec}}_G^{\varphi} \times F^n \longrightarrow E\operatorname{Vec}_G^{\varphi}$ and from the fact that the two vector bundles involved share the same dimension. This completes the proof of Theorem 2.6.

Remark 17. We have claimed in Section 2.5 that theorems similar to Theorem 2.6 hold for G-Hilbert bundles and G-simi-Hilbert bundles. Their proofs are almost identical, the only noticeable difference being in the construction of the maps $\beta^{(N)}_{x_1,(g_0,\ldots,g_{m-1})}$: here, after using the maps $\alpha_{N,x_1,(g_0,\ldots,g_{m-1})}$, we need to use the orthonormalization process (resp. the simi-orthonormalization process) to obtain orthonormal bases (resp. simi-orthonormal bases) of the fibers of φ. This works because the process is continuous and compatible with the action of $U_n(F)$ (resp. of $\operatorname{GU}_n(F)$).

B   On the homotopy type of $(\operatorname{Vec}_G^{F,\infty})^H$

Here, we fix a Lie group G, a compact subgroup H of G, and define $\operatorname{Rep}_F(H)$ as the monoid of isomorphism classes of finite-dimensional linear representations of

H (with ground field F). Our aim is to prove Theorem 4.6 which we restate for convenience:

Theorem B.1. *Let G denote a Lie group, and H be a compact subgroup of G. Then each one of the spaces $(\mathrm{Vec}_G^{F,\infty})^H$, $(i\,\mathrm{Vec}_G^{F,\infty})^H$ and $(s\,\mathrm{Vec}_G^{F,\infty})^H$ has the homotopy type of a CW-complex.*

We will only give the details in the case of $\mathrm{Vec}_G^{F,\infty}$. The strategy is as follows: recall from Proposition 3.2 that $\mathrm{Vec}_G^{F,\infty}$ and $\mathrm{Vec}_G^{\gamma(F)}$ have the same equivariant homotopy type. In Section B.1.1, we will consider the restriction functor

$$\mathrm{res}_H : \mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H \longrightarrow \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}),$$

and, in Section B.1.3, we construct a section of it

$$\mathrm{ext}_H : \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) \longrightarrow \mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$$

with a continuous equivalence of functors between $\mathrm{ext}_H \circ \mathrm{res}_H$ and $\mathrm{id}_{\mathrm{Func}(\mathcal{E}G,\gamma(F)\text{-mod})^H}$. We will deduce that $|\,\mathrm{res}_H\,| : (\mathrm{Vec}_G^{F,\infty})^H \longrightarrow |\,\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})|$ is a homotopy equivalence. In Section B.2.2, we will prove that $\mathcal{N}(\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}))_m$ has the homotopy type of a CW-complex for every $m \in \mathbb{N}$, and it will follow that its thick realization $\|\,\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})\|$ also does (cf. Appendix A of [14]). We will also show that $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$ is a good simplicial space in the sense of Segal (cf. again [14]), deduce that $\|\,\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})\|$ is homotopy equivalent to $|\,\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})|$, and conclude that $\left(\mathrm{Vec}_G^{F,\infty}\right)^H$ has the homotopy type of a CW-complex.

B.1 A homotopy equivalence from $(\mathrm{Vec}_G^{F,\infty})^H$ to $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$

Proposition 3.2 shows that the map $\mathrm{Vec}_G^{\gamma^{(m)}(F)} \to \mathrm{Vec}_G^{F,m}$ is an equivariant homotopy equivalence for any $m \in \mathbb{N}^* \cup \{\infty\}$. It will thus suffice to show that $(\mathrm{Vec}_G^{\gamma(F)})^H$ has the homotopy type of a CW-complex.

B.1.1 The functor $\mathrm{res}_H : \mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H \to \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$

Let $f : \mathcal{E}G \to \gamma(F)\text{-mod}$ be a continuous functor which is invariant for the action of H on $\mathcal{E}G$ by right-multiplication. Then, for all $h \in H$, $f(h) = f(1_G)$, and $\forall (h, h') \in H^2$, $f(1_G, hh') = f(h', hh') \circ f(1_G, h') = f(1_G, h) \circ f(1_G, h')$. The functor f thus induces a covariant functor

$$f_{|H} : \begin{cases} \mathcal{B}H & \longrightarrow \gamma(F)\text{-mod} \\ * & \longrightarrow f(1_G) \\ h & \longmapsto f(1_G, h). \end{cases}$$

Given a natural transformation $\alpha : f \to f'$ between two functors that are invariant by the action of H, we may consider the restriction $\alpha_{|H} : f_{|H} \to f'_{|H}$

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defined by $\alpha_{|H}(*) : f(1_G) \overset{\alpha(1_G)}{\to} f'(1_G)$. This yields a continuous functor

$$\mathrm{res}_H : \mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H \to \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}).$$

Taking geometric realizations, yields a continuous map

$$|\mathrm{res}_H| : \left(\mathrm{Vec}_G^{\gamma(F)}\right)^H \to |\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})|.$$

We will prove that it is a homotopy equivalence.

B.1.2 A decomposition of $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$

From now on, we set $J = \mathrm{Rep}_F(H)$. To every object f of $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$ is
assigned the linear representation $\begin{cases} H & \to f(*) \\ h & \mapsto f(h). \end{cases}$ For any $j \in J$, we let $\mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod})$
denote the full subcategory of $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$ whose objects are the functors whose corresponding linear representation of H has j as its isomorphism class.

Proposition B.2. *We have the following decomposition of topological categories:*

$$\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) = \coprod_{j \in J} \mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod}).$$

Proof. If we consider a morphism $f \overset{\alpha}{\to} f'$ in $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$, then α induces an isomorphism between the representations respectively associated to f and f'. Therefore $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$ and $\coprod_{j \in J} \mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod})$ are isomorphic as categories. We now need to prove that, for all $j \in J$, the category $\mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod})$ is open in $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$, and it suffices to show it for the spaces of objects.
We start by pointing out that

$$\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) = \mathrm{Func}\left(\mathcal{B}H, \coprod_{n \in \mathbb{N}} (\gamma_n(F)\text{-mod})\right) = \coprod_{n \in \mathbb{N}} \mathrm{Func}(\mathcal{B}H, \gamma_n(F)\text{-mod}).$$

Let $f : \mathcal{B}H \to \gamma_n(F)\text{-mod}$ be a continuous functor and define

$$\chi_f : \begin{cases} H & \longrightarrow F \\ h & \longmapsto \mathrm{Tr}(f(h)) \end{cases}$$

as the character associated to f.
We then check that the following map is continuous:

$$\chi^{(n)} : \begin{cases} \mathrm{Ob}(\mathrm{Func}(\mathcal{B}H, \gamma_n(F)\text{-mod})) & \longrightarrow L^2(H) \\ f & \longmapsto \chi_f \end{cases}$$

By construction, $\gamma_n(F)$-mod $= \varinjlim_{m \in \mathbb{N}}(\gamma_n^m(F)$-mod$)$. It thus suffices to show that
the restriction of $\chi^{(n)}$ to $\mathrm{Ob}(\mathrm{Func}(\mathcal{B}H, \gamma_n^m(F)$-mod$))$ is continuous for any $m \in \mathbb{N}$. Let f be an object of $\mathrm{Func}(\mathcal{B}H, \gamma_n^m(F)$-mod$)$ and let U_f denote the set of objects f' of $\mathrm{Func}(\mathcal{B}H, \gamma_n^m(F)$-mod$)$ for which $f'(*) \cap f(*)^{\perp} = \{0\}$. We choose an isomorphism $\varphi : f(*) \stackrel{\cong}{\to} F^n$ and notice that $U_{f(*)}$ is an open neighborhood of f in $\mathrm{Ob}(\mathrm{Func}(\mathcal{B}H, \gamma_n^m(F)$-mod$))$.

We then remark that the restriction of $\chi^{(n)}$ to $U_{f(*)}$ is the composite of the map

$$\begin{cases} U_{f(*)} & \longrightarrow \mathrm{Hom}(H, \mathrm{GL}_n(F)) \\ f' & \longmapsto \left[h \mapsto \left(\varphi \circ \pi_{f(*)}^{f'(*)} \right) \circ f'(h) \circ \left(\varphi \circ \pi_{f(*)}^{f'(*)} \right)^{-1} \right], \end{cases}$$

and of the continuous map $\mathrm{Hom}(H, \mathrm{GL}_n(F)) \longrightarrow L^2(H)$ induced by composing with the trace map $\mathrm{Tr} : M_n(F) \to F$, both of which are easily shown to be continuous. This shows that $\chi^{(n)}$ is continuous.

By the theory of linear representations of compact groups (cf. [1]), two objects of $\mathrm{Func}(\mathcal{B}H, \gamma_n(F)$-mod$)$ have the same image by $\chi^{(n)}$ if and only if their associated representations are isomorphic. Also, the image of $\chi^{(n)}$ is a discrete subset of $L^2(H)$. This proves that $\mathrm{Func}(\mathcal{B}H, \gamma_n(F)$-mod$) = \coprod_{j \in J} \mathrm{Func}_j(\mathcal{B}H, \gamma_n(F)$-mod$)$. $\qquad\square$

B.1.3 The construction of $\mathrm{ext}_H : \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) \to \mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$

Given a continuous functor $f : \mathcal{B}H \to \gamma(F)$-mod, we wish to find an H-invariant continuous functor $\tilde{f} : \mathcal{E}G \to \gamma(F)$-mod whose restriction is f (by the preceding construction), and to do this in a continuous way with respect to f. Here is the strategy: we let $V : H \to \mathrm{GL}_n(F)$ denote a homomorphism which is isomorphic to the linear representation of H associated to f, and we consider only functors $f' : \mathcal{B}H \to \gamma(F)$-mod whose associated linear representation of H is isomorphic to V. In Step 1, every such f' is considered as being associated to $\begin{cases} H & \longrightarrow B_n(F^{(\infty)}) \\ h & \longmapsto f'(h)[\mathbf{B}'] = \mathbf{B}'.V(h) \end{cases}$ for some basis $\mathbf{B}'$ of $f'(*)$.

Then we extend the map $\begin{cases} H \times B_n(F^{(\infty)}) & \longrightarrow B_n(F^{(\infty)}) \\ (h, \mathbf{B}) & \longmapsto \mathbf{B}.V(h) \end{cases}$ to a continuous map $G \times B_n(F^{(\infty)}) \longrightarrow B_n(F^{(\infty)})$ which is equivariant for a certain group action. We use this last map to assign an H-invariant continuous functor $\mathcal{E}G \to \gamma(F)$-mod to every f'. In Step 2, we prove that this construction is continuous with respect to f', by proving that the choice of $\mathbf{B}'$ may locally be made continuous with respect to f. This construction is extended to morphisms in Step 3, where the claimed properties are checked.

Step 1: Constructing $\mathrm{ext}_H^V : \mathrm{Ob}(\mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod}) \to \mathrm{Ob}(\mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod}))$
Let $j \in J$ and $V : H \to \mathrm{GL}_n(F)$ be a linear representation of H with isomor-

phism class j. We let $\mathrm{GL}_V(F)$ denote the centralizer of $\mathrm{Im}\, V$ in $\mathrm{GL}_n(F)$: this is a closed subgroup of $\mathrm{GL}_n(F)$. We set $X_j := \mathrm{Ob}(\mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod}))$ we let $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$ denote the space consisting of the linear representations of H that are isomorphic to V. Finally, recall the decomposition $\mathrm{Hom}(H, \mathrm{GL}_n(F)) = \coprod_{j_0 \in J} \mathrm{Hom}_{j_0}(H, \mathrm{GL}_n(F))$.

For the canonical right-action of H on G, the pair (G, H) is a relative H-CW-complex. The action of $\mathrm{GL}_n(F)$ on $B_n(F^{(\infty)})$ induces a right-action of $\mathrm{GL}_V(F)$ on $B_n(F^{(\infty)})$. Moreover, for every $k \in \mathbb{N}^*$, $B_n(F^k)$ has a structure of smooth manifold such that $B_n(F^{k-1})$ is a closed smooth submanifold; the action of $\mathrm{GL}_V(F)$ on $B_n(F^k)$ is free (and therefore proper), smooth, and stabilizes $B_n(F^{k-1})$. We deduce from Theorems I and II of [6] that $(B_n(F^k), B_n(F^{k-1}))$ is a relative $\mathrm{GL}_V(F)$-CW-complex. Therefore, for every $k \in \mathbb{N}^*$, the pair $\big(G \times B_n(F^k), (H \times B_n(F^k)) \cup (G \times B_n(F^{k-1}))\big)$ is a relative $(H \times \mathrm{GL}_V(F))$-CW-complex.

The action of $H \times \mathrm{GL}_V(F)$ on $G \times B_n(F^j)$ is free. Finally, we may consider the right-action of H on $B_n(F^{(\infty)})$ defined by $\mathbf{B}.h = \mathbf{B}.V(h)$ for every $h \in H$ and every $\mathbf{B} \in B_n(F^{(\infty)})$. This action is compatible with the right-action of $\mathrm{GL}_V(F)$. This yields a right-action of $H \times \mathrm{GL}_V(F)$ on $B_n(F^{(\infty)})$.

Lemma B.3. *The map* $(1_G, x) \mapsto x$ *on* $\{1_G\} \times B_n(F^{(\infty)})$ *may be extended to an* $(H \times \mathrm{GL}_V(F))$-*map:*

$$\gamma_V : G \times B_n(F^{(\infty)}) \longrightarrow B_n(F^{(\infty)}).$$

Proof. We use an induction process. Assume that, for some $k \in \mathbb{N} \setminus \{0, 1\}$, we have an equivariant map $\gamma_V^{k-1} : G \times B_n(F^{k-1}) \longrightarrow B_n(F^{(\infty)})$ such that $\forall x \in B_n(F^{k-1})$, $\gamma_V^{k-1}(1_G, x) = x$. The action of H on $B_n(F^{(\infty)})$, together with the inclusion of $B_n(F^k)$ into $B_n(F^{(\infty)})$, define an $(H \times \mathrm{GL}_V(F))$-map:

$$\beta_V^k : \begin{cases} H \times B_n(F^k) & \longrightarrow B_n(F^{(\infty)}) \\ (h, \mathbf{B}) & \longmapsto \mathbf{B}.V(h). \end{cases}$$

This yields an $(H \times \mathrm{GL}_V(F))$-map:

$$(H \times B_n(F^k)) \cup (G \times B_n(F^{k-1})) \xrightarrow{\beta_V^k \cup \gamma_V^{k-1}} B_n(F^{(\infty)}),$$

which we want to extend to $G \times B_n(F^k)$. However, $H \times \mathrm{GL}_V(F)$ acts freely on $G \times B_n(F^k)$, and the projection $B_n(F^{(\infty)}) \to *$ is a homotopy equivalence. We deduce that the map $\beta_V^k \cup \gamma_V^{k-1}$ extends to an $(H \times \mathrm{GL}_V(F))$-map $\gamma_V^k : G \times B_n(F^k) \longrightarrow B_n(F^{(\infty)})$ which has the required property. Therefore, the γ_V^j's yield an $(H \times \mathrm{GL}_V(F))$-map $\gamma_V : G \times B_n(F^{(\infty)}) \to B_n(F^{(\infty)})$. $\square$

Let $f \in X_j$ and $\mathbf{B}_f$ a basis such that $\varphi_V(\mathbf{B}_f) = f$. We set

$$\mathrm{ext}_H^V(f)(1_G, g) : \mathbf{B}_f \longmapsto \gamma_V(g, \mathbf{B}_f).$$

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This definition does not depend on the choice of $\mathbf{B}_f$. Indeed, if $\mathbf{B}'_f$ is another possible basis, there exists an $M \in \mathrm{GL}_V(F)$ such that $\mathbf{B}'_f = \mathbf{B}_f.M$, and so $\gamma_V(g, \mathbf{B}'_f) = \gamma_V(g, \mathbf{B}_f).M$ (since γ_V is an $(H \times \mathrm{GL}_V(F))$-map). Therefore, the same linear isomorphism maps $\mathbf{B}_f$ to $\gamma_V(g, \mathbf{B}_f)$, and $\mathbf{B}'_f$ to $\gamma_V(g, \mathbf{B}'_f)$. The map $\mathrm{ext}_H^V(f)$ can then easily be extended to a continuous functor $\mathcal{E}G \to \gamma(F)$-mod.

For any $h \in H$, $\gamma_V(h, \mathbf{B}_f) = \mathbf{B}_f.h = \mathbf{B}_f.V(h) = f(h)[\mathbf{B}]$, and so $\mathrm{ext}_H^V(f)(1_G, h) = f(h)$. This proves that $\mathrm{res}_H(\mathrm{ext}_H^V(f)) = f$.

For any $h \in H$ and $g \in G$, $\gamma_V(gh, \mathbf{B}_f) = \gamma_V(g, \mathbf{B}_f).h$, and

$$\mathrm{ext}_H^V(f)(1_G, g) \circ \mathrm{ext}_H^V(f)(1_G, h) : \mathbf{B}_f \longmapsto \mathbf{B}_f.h \longmapsto \gamma_V(g, \mathbf{B}_f).h = \gamma_V(gh, \mathbf{B}_f).$$

Therefore $\mathrm{ext}_H^V(f)(1_G, gh) = \mathrm{ext}_H^V(f)(1_G, g) \circ \mathrm{ext}_H^V(f)(1_G, h)$. This proves that $\mathrm{ext}_H^V(f) : \mathcal{E}G \longrightarrow \gamma(F)$-mod is invariant by the right-action of H.

Step 2: Continuity of $\mathrm{ext}_H^V : X_j \to \mathrm{Ob}(\mathrm{Func}(\mathcal{E}G, \gamma(F)\textbf{-mod}))$

The action of $\mathrm{GL}_n(F)$ on $\mathrm{Hom}(H, \mathrm{GL}_n(F))$ by conjugation is continuous; the orbit of V is $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$ and its isotropy subgroup $\mathrm{GL}_V(F)$. This yields a continuous bijection

$$\alpha_V : \begin{cases} \mathrm{GL}_n(F)/\mathrm{GL}_V(F) & \longrightarrow \mathrm{Hom}_j(H, \mathrm{GL}_n(F)) \\ [\varphi] & \longmapsto \varphi \circ V \circ \varphi^{-1}. \end{cases}$$

By Theorem 2.3.2 of [10], α_V is a homeomorphism since $\mathrm{Hom}_i(H, \mathrm{GL}_n(F))$ is locally compact (cf. [3]) and $\mathrm{GL}_n(F)$ is a σ-compact Lie group.

Recall that $B_n(F^{(\infty)})$ denotes the limit of the sequence $B_n(F^j) \to B_n(F^{j+1}) \to \dots$, and that the canonical projection $\tilde{\gamma_n} : B_n(F^{(\infty)}) \to G_n(F^{(\infty)})$ defines a $\mathrm{GL}_n(F)$-principal bundle. Let $\mathbf{B} \in B_n(F^{(\infty)})$. To $\mathbf{B}$ may be assigned a continuous map:

$$\varphi_V(\mathbf{B}) : \begin{cases} H & \to \mathrm{GL}(\mathrm{Vect}_F(\mathbf{B})) \\ h & \longmapsto [\mathbf{B} \mapsto \mathbf{B}.V(h)] \end{cases}$$

which is simply the conjugate of V by the unique isomorphism from F^n to $\mathrm{Vect}_F(\mathbf{B})$ which maps the canonical basis of F^n to $\mathbf{B}$. In particular, $\varphi_V(\mathbf{B})$ is a linear representation of H and is isomorphic to V. Also, for every $\mathbf{B}' \in B_n(F^{(\infty)})$, we have $\varphi_V(\psi(\mathbf{B})) = \psi \circ \varphi_V(\mathbf{B}) \circ \psi^{-1}$, where $\psi : \mathbf{B} \mapsto \mathbf{B}'$.

Lemma B.4. *The map* $\varphi_V : \begin{cases} B_n(F^{(\infty)}) & \longmapsto X_j \\ \mathbf{B} & \longmapsto \varphi_V(\mathbf{B}) \end{cases}$ *is a* $\mathrm{GL}_V(F)$-*principal bundle.*

Proof. Notice first that φ_V is continuous since it is the composite of
$\begin{cases} B_n(F^{(\infty)}) & \longrightarrow \mathrm{Hom}(\gamma(F)\text{-frame})^H \\ \mathbf{B} & \longmapsto [h \mapsto (\mathbf{B}, \mathbf{B}.V(h))] \end{cases}$ with $\mathrm{Hom}(\gamma(F)\text{-frame})^H \to \mathrm{Hom}(\gamma(F)\text{-mod})^H$
induced by composition of the canonical functor $\gamma(F)$-frame $\to \gamma(F)$-mod.

Given $f \in X_j$, there is a linear isomorphism $\psi : F^n \mapsto f(*)$ such that the representation associated to f is the conjugate of V by ψ. If we let $\mathbf{B}$ denote

the image of the canonical basis of F^n by ψ, then $f = \varphi_V(\mathbf{B})$. This shows φ_V is onto.

Let $\mathbf{B} \in B_n(F^{(\infty)})$ and $M \in \mathrm{GL}_V(F)$. Then M commutes with $V(h)$ for every $h \in H$ and it follows that $\varphi_V(\mathbf{B}.M) = \varphi_V(\mathbf{B})$. If $\mathbf{B}$ and $\mathbf{B}'$ have the same image by φ, then the unique $M \in \mathrm{GL}_n(F)$ such that $\mathbf{B}.M = \mathbf{B}'$ commutes with $V(h)$ for every $h \in H$, and therefore belongs to $\mathrm{GL}_V(F)$. We deduce that the fiber of φ_V over any $f \in X_j$ is isomorphic to $\mathrm{GL}_V(F)$. We now simply need to find local trivializations of φ_V, and in order to do so, it suffices to construct local sections since $B_n(F^{(\infty)}) \to G_n(F^{(\infty)})$ is a $\mathrm{GL}_n(F)$-principal bundle.

Let $f \in X_j$ and $\mathbf{B} \in B_n(F^{(\infty)})$ be a basis such that $\varphi_V(\mathbf{B}) = f$. Set $X_j^f :=$ $\left\{g \in X_j : g(*) = f(*)\right\}$. The projection: $\begin{cases} \mathrm{GL}_n(F) & \longrightarrow X_j^f \\ M & \longmapsto \varphi_V(\mathbf{B}.M) \end{cases}$ then defines a $\mathrm{GL}_V(F)$-principal bundle. Indeed, if $\varphi_V(\mathbf{B}) = f$, the linear isomorphism from F^n to $f(*)$ which maps the canonical basis of F^n to $\mathbf{B}$ induces an isomorphism from X_j^f to $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$, and we may then use the fact that α_V is a homeomorphism and that $\mathrm{GL}_n(F) \to \mathrm{GL}_n(F)/\mathrm{GL}_V(F)$ is a $\mathrm{GL}_V(F)$-principal bundle since $\mathrm{GL}_V(F)$ is a Lie subgroup of $\mathrm{GL}_n(F)$ (cf. Theorem 4.3 of [1]).

Let $\delta : U_f \to \mathrm{GL}_n(F)$ denote a local section of the preceding $\mathrm{GL}_V(F)$-principal bundle, where U_f is an open neighborhood of $f \in X_i^f$, such that $\delta(f) = \mathbf{B}$. Also, let $\beta : V_f \to B_n(F^{(\infty)})$ be a local section of the $\mathrm{GL}_n(F)$-principal bundle $\tilde{\gamma}_n : B_n(F^{(\infty)}) \to G_n(F^{(\infty)})$, where V_f is an open neighborhood of $f(*)$ in $G_n(F^{(\infty)})$, such that $\beta(f) = \mathbf{B}$. For any $g \in X_j$ such that $g(*) \in \tilde{\gamma}_n^{-1}(V_f)$, we define $\psi_g : f(*) \to g(*)$ as the linear isomorphism which maps $\mathbf{B}$ to $\beta(g(*))$. Then $g \mapsto \psi_g$ is continuous. Set then $V_f' := \left\{g \in X_j : \psi_g^{-1} \circ g \circ \psi_g \in U_f\right\}$. Obviously, V_f' is an open neighborhood of f in X_V, and we have a continuous map:

$$\begin{cases} V_f' & \longrightarrow B_n(F^{(\infty)}) \\ g & \longmapsto \psi_g\left[\mathbf{B}.\delta(\psi_g^{-1} \circ g \circ \psi_g)\right]. \end{cases}$$

This is a local section of φ_V. Indeed, let $g \in V_{f'}$ and $g_1 = \psi_g^{-1} \circ g \circ \psi_g$; then

$$\varphi_V[\psi_g(\mathbf{B}.\delta(g_1))] = \psi_g \circ \varphi_V[\mathbf{B}.\delta(g_1)] \circ \psi_g^{-1} = \psi_g \circ g_1 \circ \psi_g^{-1} = g.$$

We conclude that φ_V is a $\mathrm{GL}_V(F)$-principal bundle. $\qquad\square$

In order to prove that the map $\mathrm{ext}_H^V : X_j \to \mathrm{Func}(\mathcal{E}G, \mathcal{F})^H$ is continuous, it suffices to prove that the continuity of the map:

$$\begin{cases} G \times X_j & \longrightarrow \mathrm{Hom}(\gamma(F)\text{-mod}) \\ (g, f) & \longmapsto \mathrm{ext}_H^V(f)(1_G, g). \end{cases}$$

Let $f \in X_j$. We choose an open neighborhood U of f in X_i together with a local section $s : U \to B_n(F^{(\infty)})$ of φ_V. Composing of the maps $G \times U \xrightarrow{\mathrm{id}_G \times s} G \times$

$B_n(F^{(\infty)})$, $G \times B_n(F^{(\infty)}) \xrightarrow{(\pi_2, \gamma_V)} B_n(F^{(\infty)}) \times B_n(F^{(\infty)})$ (where π_2 denotes the projection on the second factor), and

$$\begin{cases} B_n(F^{(\infty)}) \times B_n(F^{(\infty)}) & \longrightarrow \operatorname{Hom}(\gamma(F)\text{-mod}) \\ (\mathbf{B}, \mathbf{B}') & \longmapsto [\mathbf{B} \to \mathbf{B}'] \end{cases}$$

yields

$$\begin{cases} G \times U & \longrightarrow \operatorname{Hom}(\gamma(F)\text{-mod}) \\ (g, f) & \longmapsto \operatorname{ext}_H^V(f)(1_G, g). \end{cases}$$

Since all three maps are continuous, their composite is continuous, hence so is ext_H^V.

Step 3: The functor ext_H

For every $j \in J$, we choose a representative V, and, since $\operatorname{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) = \coprod_{j \in J} \operatorname{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod})$, we finally obtain a continuous map

$$\operatorname{ext}_H : \operatorname{Ob}(\operatorname{Func}(\mathcal{B}H, \gamma(F)\text{-mod})) \longrightarrow \operatorname{Ob}(\operatorname{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H)$$

such that $\operatorname{res}_H(\operatorname{ext}_H(f)) = f$ for all $f \in \operatorname{Ob}(\operatorname{Func}(\mathcal{B}H, \gamma(F)\text{-mod}))$. We wish to extend this map to a functor $\operatorname{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) \longrightarrow \operatorname{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$. This is based on the following lemma.
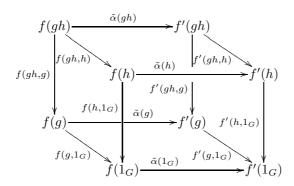
Lemma B.5. *Let $(f, f') \in \operatorname{Ob}(\operatorname{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H)^2$ together with a morphism $\alpha : \operatorname{res}_H(f) \to \operatorname{res}_H(f')$. Then there is a unique morphism $\tilde{\alpha} : f \to f'$ in $\operatorname{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$ such that $\alpha = \operatorname{res}_H(\tilde{\alpha})$.*

Proof. For all $g \in G$, we define $\tilde{\alpha}(g) : f(g) \to f'(g)$ as the unique linear isomorphism which makes the square

$$\begin{array}{ccc} f(g) & \xrightarrow{\tilde{\alpha}(g)} & f'(g) \\ {\scriptstyle f(g, 1_G)} \downarrow & & \downarrow {\scriptstyle f'(g, 1_G)} \\ f(1_G) & \xrightarrow[\alpha(*)]{} & f'(1_G) \end{array}$$

commute. By definition, $\tilde{\alpha}$ is the unique morphism from f to f' in $\operatorname{Func}(\mathcal{E}G, \gamma(F)\text{-mod})$. We only need to prove that it is invariant under the action of H.

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Let then $(g, h) \in G \times H$, and consider the commutative diagram:

$$
\begin{array}{ccc}
f(gh) & \xrightarrow{\tilde{\alpha}(gh)} & f'(gh) \\
\end{array}
$$

Since α is a morphism from $\mathrm{res}_H(f)$ to $\mathrm{res}_H(f')$, we deduce that $\tilde{\alpha}(h) = \tilde{\alpha}(1_G) = \alpha(*)$. Since f and f' are invariant under the action of H, we have $f(gh, h) = f(g, 1_G)$, and $f'(gh, h) = f'(g, 1_G)$. We deduce that $\tilde{\alpha}(gh) = \tilde{\alpha}(g)$. $\qquad\square$

Given a morphism $\alpha : f \to f'$ in $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$, we now define $\mathrm{ext}_H(\alpha)$ as the morphism in $\mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$ which is associated to $\mathrm{ext}_H(f)$, $\mathrm{ext}_H(f')$ and α as in Lemma B.5. That ext_H defines a functor is then obvious from the uniqueness in Lemma B.5. That ext_H is continuous on morphisms is also obvious, and we obtain a continuous functor

$$\mathrm{ext}_H : \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod}) \longrightarrow \mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H.$$

Obviously

$$\mathrm{res}_H \circ \mathrm{ext}_H = \mathrm{id}_{\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})}.$$

It remains to prove that $|\,\mathrm{ext}_H\,| \circ |\,\mathrm{res}_H\,|$ is homotopic to the identity map of $|\,\mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H\,|$. Let f be an object of $\mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$. Then $\mathrm{res}_H(f) = \mathrm{res}_H((\mathrm{ext}_H \circ \mathrm{res}_H)(f))$, and we can thus consider the morphism $f \to \mathrm{ext}_H(\mathrm{res}_H(f))$ of $\mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H$ which is associated to f, $\mathrm{ext}_H(\mathrm{res}_H(f))$ and $\mathrm{id}_{f(1_G)}$ by Lemma B.5. This defines a continuous natural transformation

$$\eta : \mathrm{id}_{\mathrm{Func}(\mathcal{E}G, \gamma(F)\text{-mod})^H} \longrightarrow (\mathrm{ext}_H \circ \mathrm{res}_H)(f).$$

Therefore $|\,\mathrm{ext}_H\,|$ is a homotopy inverse of $|\,\mathrm{res}_H\,|$.

Conclusion: The space $(\mathrm{Vec}_G^{F, \infty})^H$ has the homotopy type of $|\,\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})\,|$.

Remarks 18. (i) The previous constructions can also be achieved in the cases of $i\,\mathrm{Vec}_G^{F, \infty}$ and $s\,\mathrm{Vec}_G^{F, \infty}$. The only difference is that we need to consider Hilbert representations of H. Proposition B.2 obviously implies a similar result in these cases, since the topological categories $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-imod})$ and $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-smod})$ may be seen as embedded in $\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})$ (and since two Hilbert representations of H are isomorphic as linear representations of H if and only if they are isomorphic as Hilbert representations

of H). The construction of ext_H is then achieved by replacing $\mathrm{GL}_n(F)$ by $U_n(F)$ (resp. by $\mathrm{GU}_n(F)$), and free n-tuples by orthonormal n-tuples (resp. simi-orthonormal n-tuples). We deduce that $(i\,\mathrm{Vec}_G^{F,\infty})^H$ is homotopy equivalent to $|\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-imod})|$, and that $(s\,\mathrm{Vec}_G^{F,\infty})^H$ is homotopy equivalent to $|\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-smod})|$. Interestingly, we may prove directly from there that $(i\,\mathrm{Vec}_G^{F,\infty})^H$ is homotopy equivalent to $(s\,\mathrm{Vec}_G^{F,\infty})^H$. Indeed, any continuous functor from $\mathcal{B}H$ to $\gamma(F)$-smod induces a functor from $\mathcal{B}H$ to $\gamma(F)$-imod, since H is compact. We thus define a functor

$$\begin{cases} \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-smod}) & \longrightarrow \mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-imod}) \\ f & \longmapsto f \\ \alpha : (f \to f') & \longmapsto \frac{\alpha}{\|\alpha\|} : (f \to f'), \end{cases}$$

and it is easy to check that it induces a homotopy inverse of the inclusion $|\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-imod})| \subset |\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-smod})|$.

(ii) When G is discrete, the previous construction may adapted to any of the spaces $\mathrm{Vec}_G^{F,m}$, for $m \in \mathbb{N}^*$. By Lemma B.5, it suffices to construct a section of the restriction map on objects. For every class z in G/H, we choose an element $g_z \in G$ such that $[g_z] = z$. Let then $f : \mathcal{B}H \to \gamma^{(m)}(F)$-mod. We define a functor $\tilde{f} : \mathcal{E}G \to \gamma^{(m)}(F)$-mod by $\tilde{f}(1_G, g_z.h) := f(h)$ for all $z \in G/H$ and $h \in H$. We can easily check that this defines an H-invariant functor, that the map $f \mapsto \tilde{f}$ is continuous, and that $\mathrm{res}_H(\tilde{f}) = f$. This proves that $(\mathrm{Vec}_G^{F,m})^H$ is homotopy equivalent to $|\mathrm{Func}(\mathcal{B}H, \gamma^{(m)}(F)\text{-mod})|$ for all $m \in \mathbb{N}$, whenever G is discrete. The same line of reasoning also applies to $i\,\mathrm{Vec}_G^{F,m}$ and $s\,\mathrm{Vec}_G^{F,m}$.

B.2 On the homotopy type of $|\mathrm{Func}(\mathcal{B}H, \gamma(F)\text{-mod})|$

In this section, we will prove the following result.

Proposition B.6. *Let $j \in \mathrm{Rep}_F(H)$ be an n-dimensional class. Then $|\mathrm{Func}_j(\mathcal{B}H, \gamma(F)\text{-mod})|$ has the homotopy type of a CW-complex.*

For any $k \in \mathbb{N}$, we set $\mathcal{C}_{j,k} := \mathrm{Func}_j(\mathcal{B}H, \gamma_n^k(F)\text{-mod})$ and $\mathcal{C}_j := \mathrm{Func}_j(\mathcal{B}H, \gamma_n(F)\text{-mod})$. Clearly, $\mathcal{C}_j = \varinjlim_{k \in \mathbb{N}} (\mathcal{C}_{j,k})$.

We will show that the nerve $(\mathcal{N}(\mathcal{C}_j)_m)_{m \in \mathbb{N}}$ of $\mathcal{C}_j$ is a good simplicial space (cf. Appendix A of [14]) and that each of its components $\mathcal{N}(\mathcal{C}_j)_m$ has the homotopy type of a CW-complex. To do so, we will equip the space $\mathcal{N}(\mathcal{C}_{j,k})_m$ with a structure of smooth manifold, for every pair $(k, m) \in \mathbb{N}^* \times \mathbb{N}$, and prove that these structures are compatible with standard inclusions.

B.2.1 A structure of smooth manifold on $\mathcal{N}(\gamma_n^k(F)\text{-mod})_m$

We start with a short reminder of the canonical manifold structure on $G_n(F^k)$. For every $x \in G_n(F^k)$, let U_x denote the subspace of $G_n(F^k)$ consisting of those

elements y such that $y \cap x^\perp = \{0\}$ and notice that U_x is an open neighborhood of x in $G_n(F^k)$. For any pair (y, z) of subspaces of F^k, let π_z^y denote the restriction to y of the orthogonal projection on z. In the case $y = F^k$, we set $\pi_z := \pi_z^{F^k}$.

We then obtain a chart $\psi_x : \begin{cases} U_x & \stackrel{\cong}{\longrightarrow} L(x, x^\perp) \\ y & \longmapsto \pi_{x^\perp} \circ (\pi_x^y)^{-1}. \end{cases}$ The chart transitions are smooth.

For every $x \in G_n(F^k)$, we choose an isomorphism $\alpha_x : F^n \stackrel{\cong}{\to} x$ and set

$$\varphi_x : \begin{cases} U_x \times \mathrm{GL}_n(F) & \longrightarrow \tilde{\gamma}_n^k(F)^{-1}(U_x) \\ (y, \mathbf{B}) & \longmapsto \pi_y(\alpha_x(\mathbf{B})). \end{cases}$$

Clearly, φ_x is a homeomorphism over U_x, whilst $(U_x, \psi_x)_{x \in G_n(F^k)}$ is a system of local trivializations of $\gamma_n^k(F)$, and the trivialization transitions are smooth. We have just defined a structure of smooth $\mathrm{GL}_n(F)$-principal bundle on $\tilde{\gamma}_n^k(F)$.

For $(x_0, \ldots, x_m) \in G_n(F^k)^{m+1}$, let $U_{x_0, \ldots, x_m}^{(k)}$ denote the set consisting of those m-simplices $y_0 \to \cdots \to y_m$ in $\mathcal{N}(\gamma_n^k\text{-mod})_m$ such that $y_i \in U_{x_i}$ for all $i \in \{0, \ldots, m\}$. The family $(U_{x_0, \ldots, x_m}^{(k)})_{(x_0, \ldots, x_m) \in G_n(F^k)^{m+1}}$ is an open cover of $\mathcal{N}(\gamma_n^k(F)\text{-mod})_k$. We then obtain charts

$$\psi_{x_0, \ldots, x_m}^{(k)} : \begin{cases} U_{x_0, \ldots, x_m}^{(k)} & \longrightarrow \left[\prod_{j=0}^m L(x_j, x_j^\perp) \right] \times (\mathrm{GL}_n(F))^m \\ (y_0 \stackrel{\alpha_0}{\to} \cdots \stackrel{\alpha_{m-1}}{\to} y_m) & \longmapsto \left((\psi_{x_j}(y_j))_{0 \le j \le m}, (\alpha_{x_{j+1}}^{-1} \circ \pi_{x_{j+1}} \circ \alpha_j \circ (\pi_{x_j}^{y_j})^{-1} \circ \alpha_{x_j})_{0 \le j \le m-1} \right) \end{cases}$$

and, again, the chart transitions are smooth. We therefore end up with a structure of smooth manifold on $\mathcal{N}(\gamma_n^k(F)\text{-mod})_m$.

B.2.2 A structure of smooth manifold on $\mathcal{N}(\mathcal{C}_{j,k})_m$

For $m \in \mathbb{N}$, $\sigma \subset \{0, \ldots, m-1\}$, and a simplicial space X, we set $X_{m,\sigma} := \bigcap_{j \in \sigma} s_j^m(X_{m-1})$.

Let $V : H \to \mathrm{GL}_n(F)$ be a linear representation with isomorphism class j. We recall the homeomorphism $\alpha_V : \begin{cases} \mathrm{GL}_n(F)/\mathrm{GL}_V(F) & \longrightarrow \mathrm{Hom}_j(H, \mathrm{GL}_n(F)) \\ [M] & \longmapsto M.V.M^{-1}. \end{cases}$

There is a unique structure of smooth manifold on $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$ such that α_V is a diffeomorphism. We can check that this structure does not depend on the choice of V. To see this, we remind the reader of the following classical lemma.

Lemma B.7. *Let G be a Lie group, H a closed subgroup of G, and $g \in G$. Then the unique G-map $\beta_{G,H,g} : \begin{cases} G/H & \longrightarrow G/gHg^{-1} \\ [1_G] & \longmapsto [g] \end{cases}$ which assigns $[g]$ to $[1_G]$ is a diffeomorphism.*

Let V' be another representative of j. If $M \in \mathrm{GL}_n(F)$ is such that $V' = M.V.M^{-1}$, then $\alpha_{V'} = \alpha_V \circ \beta_{\mathrm{GL}_n(F),\mathrm{GL}_V(F),M}$, and so $\alpha_{V'}$ is a diffeomorphism if and only if α_V is a diffeomorphism.

The group $\mathrm{GL}_n(F)$ acts on the left on $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$ (by the conjugation action). This action is smooth, because it is induced by α_V and the smooth map

$$
\begin{cases}
\mathrm{GL}_n(F) \times (\mathrm{GL}_n(F)/\mathrm{GL}_V(F)) & \longrightarrow \mathrm{GL}_n(F)/\mathrm{GL}_V(F) \\
(M, [M']) & \longmapsto [M.M'] .
\end{cases}
$$

Consider the projection

$$
\pi_{j,k,m} : \begin{cases}
\mathcal{N}(\mathcal{C}_{j,k})_m & \longrightarrow \mathcal{N}(\gamma_n^k(F)\text{-mod})_m \\
\left(f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m \right) & \longmapsto \left(f_0(*) \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m(*) \right).
\end{cases}
$$

The fiber of $\pi_{j,k,m}$ over any point of $\mathcal{N}(\gamma_n^k\text{-mod})$ is clearly isomorphic to $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$. It is now easy to check that the following result is true.

Proposition B.8. *Let $k \in \mathbb{N}^*$.*
For every $(x_0, \ldots, x_m) \in G_n(F^k)^{m+1}$, we set

$$
\Phi_{x_0,\ldots,x_m}^{(k)} : \begin{cases}
U_{x_0,\ldots,x_m}^{(k)} \times \mathrm{Hom}_j(H, GL_n(F)) & \longrightarrow \pi_{j,k,m}^{-1}(U_{x_0,\ldots,x_m}^{(k)}) \\
(y_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} y_m, f) & \longmapsto \left[(\pi_{x_0}^{y_0})^{-1} \circ \alpha_{x_0} \circ f \circ \alpha_{x_0}^{-1} \circ \pi_{x_0}^{y_0} \right] \overset{\alpha_0}{\to} \cdots
\end{cases}
$$

Then $\Phi_{x_0,\ldots,x_m}^{(k)}$ is a homeomorphism over $U_{x_0,\ldots,x_m}^{(k)}$.

Let $f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m$ be an m-simplex in $\mathcal{N}(\mathcal{C}_{j,k})_m$. For all $\ell \in \{0, \ldots, m\}$, we set $x_\ell := f_\ell(*)$, and let $x_\ell^\perp$ and y_ℓ respectively denote the orthogonal complement of x_ℓ in F^k and in F^{k+1}.
Then:

(i) $\left(\psi_{x_0,\ldots,x_m}^{(k)} \times \mathrm{id}_{\mathrm{Hom}_j(H, GL_n(F))} \right) \circ (\Phi_{x_0,\ldots,x_m}^{(k)})^{-1}$ *is a homeomorphism onto*

$$
\left[\prod_{\ell=0}^m L(x_\ell, x_\ell^\perp) \right] \times GL_n(F)^m \times \mathrm{Hom}_j(H, GL_n(F)).
$$

(ii) *The restriction of $\left(\psi_{x_0,\ldots,x_m}^{(k+1)} \times \mathrm{id}_{\mathrm{Hom}_j(H, GL_n(F))} \right) \circ (\Phi_{x_0,\ldots,x_m}^{(k+1)})^{-1}$ to $\pi_{j,k+1,m}^{-1}(U_{x_0,\ldots,x_m}^{(k+1)}) \cap \mathcal{N}(\mathcal{C}_{j,k})_m$ is precisely $\left(\psi_{x_0,\ldots,x_m}^{(k)} \times \mathrm{id}_{\mathrm{Hom}_j(H, GL_n(F))} \right) \circ (\Phi_{x_0,\ldots,x_m}^{(k)})^{-1}$.*

(iii) *If $f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m \in \mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ for some $\sigma \subset \{0, \ldots, m\}$, then $\left(\psi_{x_0,\ldots,x_m}^{(k)} \times \mathrm{id}_{\mathrm{Hom}_j(H, GL_n(F))} \right) \circ (\Phi_{x_0,\ldots,x_m}^{(k)})^{-1}$ maps $\pi_{j,k,m}^{-1}(U_{x_0,\ldots,x_m}^{(m)}) \cap \mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ to*

$$
\left\{ ((\varphi_\ell)_{0 \le \ell \le m}, (\psi_\ell)_{0 \le \ell \le m-1}) \in \left[\prod_{\ell=0}^m L(x_\ell, x_\ell^\perp) \right] \times GL_n(F)^m : \forall \ell \in \sigma, \begin{cases} \varphi_\ell = \varphi_{\ell+1} \\ \psi_\ell = I_n \end{cases} \right\}
$$
$$
\times \mathrm{Hom}_j(H, GL_n(F))
$$

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Corollary B.9. *For every $m \in \mathbb{N}$, $\mathcal{N}(\mathcal{C}_{j,k})_m$ has a structure of smooth manifold.*

Proof. The maps $\Phi^{(k)}_{x_0,\ldots,x_m}$ define a system of local trivializations. The transitions are smooth hence $\pi_{j,k,m}$ is equipped with a structure of smooth fibre bundle with fiber $\mathrm{Hom}_j(H, \mathrm{GL}_n(F))$ and structural group $\mathrm{GL}_n(F)$. $\square$

Corollary B.10. *Let $k \in \mathbb{N}$, $m \in \mathbb{N}^*$, $\sigma \subset \sigma' \subset \{0, \ldots, m-1\}$.*

- *$\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ is a closed smooth submanifold of $\mathcal{N}(\mathcal{C}_{j,k})_m$.*

- *$\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma'}$ is a closed smooth submanifold of $\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$.*

- *$\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ is a closed smooth submanifold of $\mathcal{N}(\mathcal{C}_{j,k+1})_{m,\sigma}$.*

Proof. By Proposition B.8, it suffices to proof the closeness.

Let $\ell \in \{0, \ldots, m-1\}$, and $f_0 \overset{\alpha_0}{\to} \ldots \overset{\alpha_{m-1}}{\to} f_m$ be an m-simplex in $\mathcal{N}(\mathcal{C}_{j,k})_m \smallsetminus \mathcal{N}(\mathcal{C}_{j,k})_{m,\{\ell\}}$.

If $f_\ell(*) \neq f_{\ell+1}(*)$, then we choose a pair (U, U') of disjoint open subsets of $G_n(F^k)$ such that $f_\ell(*) \in U$ and $f_{\ell+1}(*) \in U'$; then $\{(g_0 \to \cdots \to g_m) \in \mathcal{N}(\mathcal{C}_{j,k})_m : g_\ell(*) \in U, g_{\ell+1}(*) \in U'\}$ is an open neighborhood of $f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m$ in $\mathcal{N}(\mathcal{C}_{j,k})_m$ which intersects $\mathcal{N}(\mathcal{C}_{j,k})_{m,\{\ell\}}$ trivially.

If $f_\ell(*) = f_{\ell+1}(*)$, then

$$(\psi^{(k)}_{x_0,\ldots,x_m} \circ \pi_{j,k,m})^{-1} \left(\left[\prod_{l=0}^m L(x_l, x_l^\perp) \right] \times \mathrm{GL}_n(F)^\ell \times (\mathrm{GL}_n(F) \smallsetminus \{I_n\}) \times \mathrm{GL}_n(F)^{m-\ell-1} \right)$$

is an open neighborhood of $f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m$ in $\mathcal{N}(\mathcal{C}_{j,k})_m$ which intersects $\mathcal{N}(\mathcal{C}_{j,k})_{m,\{\ell\}}$ trivially. This proves that $\mathcal{N}(\mathcal{C}_{j,k})_{m,\{\ell\}}$ is a closed subspace of $\mathcal{N}(\mathcal{C}_{j,k})_m$. We deduce that $\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma'}$ is a closed subspace of $\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ whenever $\sigma \subset \sigma' \subset \{0, \ldots, m-1\}$.

Let $\sigma \subset \{0, \ldots, m-1\}$, and $f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m$ be an m-simplex in $\mathcal{N}(\mathcal{C}_{j,k+1})_{m,\sigma} \smallsetminus \mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$. Then there exists an index $\ell \in \{0, \ldots, m\}$ such that $f_\ell(*) \in G_n(F^{k+1}) \smallsetminus G_n(F^k)$. The subset of $\mathcal{N}(\gamma_n^{k+1}(F)\text{-mod})_m$ consisting of those m-simplices $y_0 \to \cdots \to y_m$ such that $y_\ell \in G_n(F^{k+1}) \smallsetminus G_n(F^k)$, is open in $\mathcal{N}(\gamma_n^{k+1}(F)\text{-mod})_m$, and its inverse image by $\pi_{j,k+1,m}$ is an open neighborhood of $f_0 \overset{\alpha_0}{\to} \cdots \overset{\alpha_{m-1}}{\to} f_m$ in $\mathcal{N}(\mathcal{C}_{j,k+1})_m$ which intersects $\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ trivially. It follows that $\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$ is a closed subspace of $\mathcal{N}(\mathcal{C}_{j,k+1})_{m,\sigma}$. $\square$

Corollary B.11. *For every $m \in \mathbb{N}$, $\mathcal{N}(\mathcal{C}_j)_m$ has the homotopy type of a CW-complex.*

Proof. Let $m \in \mathbb{N}$. By Proposition B.10, $\mathcal{N}(\mathcal{C}_{j,k-1})_m$ is a closed submanifold of $\mathcal{N}(\mathcal{C}_{j,k})_m$ for every $k \in \mathbb{N} \smallsetminus \{0, 1\}$. In particular, the inclusion $\mathcal{N}(\mathcal{C}_{j,k-1})_m \subset \mathcal{N}(\mathcal{C}_{j,k})_m$ is a closed cofibration, and it follows that $\mathcal{N}(\mathcal{C}_j)_m$, which is the colimit of this sequence of inclusions, has the homotopy type of the homotopy colimit of the sequence

$$\mathcal{N}(\mathcal{C}_{j,1})_m \subset \cdots \subset \mathcal{N}(\mathcal{C}_{j,k-1})_m \subset \mathcal{N}(\mathcal{C}_{j,k})_m \subset \cdots.$$

For every $k \in \mathbb{N}^*$, we equip $\mathcal{N}(\mathcal{C}_{j,k})_m$ with a CW-complex structure (this is possible since $\mathcal{N}(\mathcal{C}_{j,k})_m$ is a smooth manifold). Thus every inclusion $\mathcal{N}(\mathcal{C}_{j,k})_m \subset \mathcal{N}(\mathcal{C}_{j,k+1})_m$ is homotopic to a cellular map f_k. The homotopy colimit of the sequence

$$\mathcal{N}(\mathcal{C}_{j,1})_m \overset{f_1}{\to} \cdots \overset{f_{k-1}}{\to} \mathcal{N}(\mathcal{C}_{j,k})_m \overset{f_k}{\to} \mathcal{N}(\mathcal{C}_{j,k+1})_m \subset \cdots$$

is then a CW-complex and has the homotopy type of the previous homotopy colimit. It follows that $\mathcal{N}(\mathcal{C}_j)_m$ has the homotopy type of a CW-complex. $\square$

Corollary B.12. *Let $m \in \mathbb{N}$ and $\sigma \subset \sigma' \subset \{0, \ldots, m\}$.*
Then $\mathcal{N}(\mathcal{C}_j)_{m,\sigma'} \hookrightarrow \mathcal{N}(\mathcal{C}_j)_{m,\sigma}$ is a closed cofibration.

Proof. We notice that $\mathcal{N}(\mathcal{C}_j)_{m,\sigma'} = \varinjlim_{k \in \mathbb{N}^*} \mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma'}$ and that $\mathcal{N}(\mathcal{C}_j)_{m,\sigma} = \varinjlim_{k \in \mathbb{N}^*} \mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma}$.
Furthermore, for every $k \in \mathbb{N}^*$, $\mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma} \cap \mathcal{N}(\mathcal{C}_{j,k+1})_{m,\sigma'} = \mathcal{N}(\mathcal{C}_{j,k})_{m,\sigma'}$. The corollary is thus derived from Proposition B.10, from the fact that the inclusion of a closed smooth submanifold is a closed cofibration, and from the following lemma (the proof of which is straightforward). $\square$

Lemma B.13. *Let*

$$
\begin{array}{ccccccccccc}
A_0 & \overset{\alpha_0}{\longrightarrow} & A_1 & \longrightarrow & \cdots & \overset{\alpha_{n-1}}{\longrightarrow} & A_n & \overset{\alpha_n}{\longrightarrow} & A_{n+1} & \overset{\alpha_{n+1}}{\longrightarrow} & \cdots \\
f_0 \downarrow & & f_1 \downarrow & & & & f_n \downarrow & & f_{n+1} \downarrow & & \\
B_0 & \overset{\beta_0}{\longrightarrow} & B_1 & \longrightarrow & \cdots & \overset{\beta_{n-1}}{\longrightarrow} & B_n & \overset{\beta_n}{\longrightarrow} & B_{n+1} & \overset{\beta_{n+1}}{\longrightarrow} & \cdots
\end{array}
$$

be a commutative diagram in the category of topological spaces such that

(i) all morphisms are closed cofibrations;

(ii) for every $n \in \mathbb{N}$, $A_{n+1} \cap B_n = A_n$, where A_n, A_{n+1} and B_n are seen as subspaces of B_{n+1}.

Then $\varinjlim_{n \in \mathbb{N}} A_n \longrightarrow \varinjlim_{n \in \mathbb{N}} B_n$ is a closed cofibration.

Proof of Proposition B.6: We deduce from Corollary B.12 that $\mathcal{C}_j$ is a good simplicial space and it follows from point (iv) of Proposition A.1 of [14] that $|\mathcal{C}_j|$ has the homotopy type of $\|\mathcal{C}_j\|$. Also, every component of the nerve of $\mathcal{C}_j$ has the homotopy type of a CW-complex, and we deduce, using point (i) of Proposition A.1 of [14], that $\|\mathcal{C}_j\|$ has the homotopy type of a CW-complex. Hence $|\mathcal{C}_j|$ has the homotopy type of a CW-complex. $\square$

From Proposition B.6, we deduce that $|\operatorname{Func}(\mathcal{B}H, \gamma(F)\text{-mod})| = \coprod_{j \in J} |\operatorname{Func}_j(\mathcal{B}\mathcal{H}, \gamma(F)\text{-mod})|$

has the homotopy type of a CW-complex, hence so does $(\operatorname{Vec}_G^{F,\infty})^H$. This completes the proof of Theorem 4.6.

Remarks 19. (i) In a similar fashion, we may prove that $|\operatorname{Func}(\mathcal{B}H, \gamma^{(m)}(F)\text{-mod})|$ has the homotopy type of a CW-complex for every $m \in \mathbb{N}^*$. Since, for $m \in \mathbb{N}^*$, we do not know how to construct an extension functor as in Section B.1.3, this is not really interesting. However, when G is discrete and H is a finite subgroup of it, this shows that $(\operatorname{Vec}_G^{F,m})^H$ has the homotopy type of a CW-complex for every $m \in \mathbb{N}^* \cup \{\infty\}$.

(ii) The proof of Proposition B.6 may easily be adapted to the case of $|\operatorname{Func}_j(\mathcal{B}\mathcal{H}, \gamma(F)\text{-imod})|$: it suffices to replace $\operatorname{GL}_n(F)$ by $U_n(F)$ everywhere. We then deduce that $(i\operatorname{Vec}_G^{F,\infty})^H$ has the homotopy type of a CW-complex, and $(s\operatorname{Vec}_G^{F,\infty})^H$ also does, since it is homotopy equivalent to it.

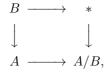# C A note on Γ-spaces

When $\underline{A}$ is a $\Gamma - G$-space, BA will denote its thick geometric realization (where a *colimit* is used instead of a *homotopy colimit*). For every finite set S, we have a $\Gamma - G$-space $\underline{A}(S \times -)$, and we let $B\underline{A}(S)$ denote its thick geometric realization. Then $B\underline{A}$ is a $\Gamma - G$-space, and $B\underline{A}(\mathbf{1}) = BA$.

It is only when $\underline{A}(\mathbf{0})$ is a point that there is a well-defined canonical map $\underline{A}(\mathbf{1}) \to \Omega BA$. It thus appears that the map Segal deals with in [14] has to be understood as the composite of the canonical map $\underline{A}(\mathbf{1}) \to \Omega(BA/\underline{A}(\mathbf{0}))$ with a homotopy inverse of $\Omega BA \to \Omega(BA/\underline{A}(\mathbf{0}))$. The existence of such a homotopy inverse relies upon the following lemma (in its non-equivariant form):

Lemma C.1. *Let G be a topological group, B be a well-pointed G-space, and $B \hookrightarrow A$ be a closed G-cofibration. Assume B is G-contractible. Then $A \to A/B$ is a pointed equivariant homotopy equivalence.*

Proof. Since $B \hookrightarrow A$ is a G-cofibration and $B \to *$ is an equivariant homotopy equivalence, we may apply the equivariant version of the homotopy theorem for cofibrations to the push-out square

$$
\begin{array}{ccc}
B & \longrightarrow & * \\
\downarrow & & \downarrow \\
A & \longrightarrow & A/B,
\end{array}
$$

and therefore deduce that $A \to A/B$ is an equivariant homotopy equivalence. Since B is a well-pointed G-space and A/B also is - judging from the preceding push-out square - we deduce that $A \to A/B$ is a pointed equivariant homotopy equivalence. $\qquad\square$

In order to have a map of the form claimed by Segal, it then seems that we need to assume that $\underline{A}(\mathbf{0})$ is a well-pointed space. In this case $\underline{A}(\mathbf{0})$ will be both contractible and well-pointed, and we may deduce from Lemma C.1 that

$\Omega BA \to \Omega(BA/\underline{A}(\mathbf{0}))$ is a pointed homotopy equivalence. In the case of $\Gamma - G$-spaces, we assume that $\underline{A}(\mathbf{0})$ is a well-pointed G-space, and obtain the same result in the equivariant context. If we choose a pointed (equivariant) inverse $s : \Omega(BA/\underline{A}(\mathbf{0})) \to \Omega BA$ of $\Omega BA \to \Omega(BA/\underline{A}(\mathbf{0}))$ up to pointed (equivariant) homotopy, we finally obtain a map

$$\underline{A}(\mathbf{1}) \longrightarrow \Omega(BA/\underline{A}(\mathbf{0})) \longrightarrow \Omega BA$$

which has the properties claimed by Segal in [14] (again, in the equivariant context).

Remark 20. We could have simply assumed that $\underline{A}(\mathbf{0})$ is a point (as in the definition of an equivariant Γ-space given by Lück and Oliver in [9]). However, if $\underline{A}$ is a $\Gamma - G$-space defined in such a way, then $B\underline{A}(\mathbf{0})$ is not a point.

The following easy lemma makes our definition relevant :

Lemma C.2. *Let G be a topological group and $\underline{A}$ be a $\Gamma - G$-space such that $\underline{A}(\mathbf{0})$ is a well-pointed G-space. Then $B^n \underline{A}(\mathbf{0})$ is a well-pointed G-space for every positive integer n.*

Our next problem is to establish the properties of the various maps $\Omega^n B^n A \to \Omega^{n+1} B^{n+1} A$ that can be deduced from the previous ones. This is essentially treated in Lemma 2.6 of [9], but the previous remarks make a more rigorous proof necessary (part of the problem being that it is not obvious why the square appearing in the proof of the aforementioned lemma is commutative up to equivariant homotopy).

Given a $\Gamma - G$-space $\underline{A}$ such that $\underline{A}(\mathbf{0})$ is well-pointed, we construct the $\Gamma - G$-space $B^n \underline{A}$ as follows for every finite set S, we consider the G-space

$$\coprod_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \underline{A}(S \times \mathbf{i_1} \times \cdots \times \mathbf{i_n}) \times (\Delta^{i_1} \times \cdots \times \Delta^{i_n}),$$

and we then construct the quotient space using the face maps. This quotient space is no other than $B^n \underline{A}(S)$, as defined by Segal in [14], and this can easily be checked by induction on n. In particular, $B^n A$ is a quotient space of $\coprod_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \underline{A}(\mathbf{i_1} \times \cdots \times \mathbf{i_n}) \times (\Delta^{i_1} \times \cdots \times \Delta^{i_n})$, whilst $B^n \underline{A}(\mathbf{0}) = \underline{A}(\mathbf{0}) \times B^n *$ is equivariantly contractible.

For $n \geq 0$ and $0 \leq k \leq n$, the identification maps

$$\underline{A}(\mathbf{0}\times\mathbf{i_1}\times\cdots\times\mathbf{i_n})\times(\Delta^{i_1}\times\cdots\times\Delta^{i_n}) \xrightarrow{\cong} \underline{A}(\mathbf{i_1}\times\cdots\times\mathbf{i_k}\times\mathbf{0}\times\cdots\times\mathbf{i_n})\times(\Delta^{i_1}\times\ldots\Delta^{i_k}\times\Delta^0\times\cdots\times\Delta^{i_n})$$

give rise to an injection

$$j_{n+1}^k : B^n \underline{A}(\mathbf{0}) \hookrightarrow B^{n+1} A.$$

which is a closed G-cofibration. For every positive integer n, we set:

$$B_0^n A := \underset{0\leq k\leq n-1}{\cup} j_n^k(B^{n-1}\underline{A}(\mathbf{0})) \subset B^n A.$$

Proposition C.3. *Let $\underline{A}$ be a $\Gamma - G$-space. Then:*

(a) The embedding $B_0^n A \hookrightarrow B^n A$ is a closed G-cofibration.

(b) The space $B_0^n A$ is G-contractible.

Proof. We will use the notation "$*$" to denote the trivial $\Gamma - G$-space (with a point in every degree). The set $B_0^n A$ is a closed subspace of $B^n A$, since it is a finite union of closed subsets. Also, we remark that $B_0^n A \cong \underline{A}(\mathbf{0}) \times B_0^n *$. Since $\underline{A}(\mathbf{0})$ is G-contractible and G acts trivially on $B_0^n *$, statement (b) will follow if we prove that $B_0^n *$ is contractible. This last proof relies on the following double induction process.

The result is clearly true when $n = 1$ (since $B_0^1 * = *$).

Assume that there is some integer $n > 0$ such that, for every $0 \le k \le n - 1$, the space $\bigcup_{i=0}^{k} j_n^i(B^{n-1}*)$ is contractible. Then $j_{n+1}^k(B^n*) \cong B^n*$ is contractible for every integer $k \in [n]$. In particular, $j_{n+1}^0(B^n*)$ is contractible

Assume further that, for some $k \in [n-1]$, $\bigcup_{i=0}^{k} j_{n+1}^i(B^n*)$ is contractible. We can then consider the following cocartesian square:

$$
\begin{array}{ccc}
\bigcup_{i=0}^{k} j_n^i(B^{n-1}*) & \overset{\cong}{\longrightarrow} & j_{n+1}^{k+1}(B^n*) \\
\downarrow & & \downarrow \\
\bigcup_{i=0}^{k} j_{n+1}^i(B^n*) & \overset{\cong}{\longrightarrow} & \bigcup_{i=0}^{k+1} j_{n+1}^i(B^n*).
\end{array}
$$

By assumptions, the two upper spaces are contractible, and it follows that the upper horizontal map is a homotopy equivalence. Also, the left-hand vertical map is a cofibration. We can then apply the homotopy theorem for cofibrations to obtain that $\bigcup_{i=0}^{k+1} j_{n+1}^i(B^n*)$ has the homotopy type of $\bigcup_{i=0}^{k} j_{n+1}^i(B^n*)$, and deduce that it is contractible. We conclude that $B_0^n *$ is contractible for every positive integer n, which proves statement (b).

It remains to prove that $B_0^n A \hookrightarrow B^n A$ is a G-cofibration. However the maps $j_n^0, j_n^1, \ldots, j_n^{n-1}$ are all closed G-cofibrations. Moreover, for every proper $\sigma \subsetneq \{0, \ldots, n-1\}$, and every $i \in \{0, \ldots, n-1\} \smallsetminus \sigma$, the inclusion $\bigcap_{l \in \sigma} j_n^l(B^{n-1}\underline{A}(\mathbf{0})) \hookrightarrow \bigcap_{l \in \sigma \cup \{i\}} j_n^l(B^{n-1}\underline{A}(\mathbf{0}))$ is a G-cofibration (this is reduced to the easy case of the point by noticing that $B^{n-1}\underline{A}(\mathbf{0}) = \underline{A}(\mathbf{0}) \times B^{n-1}*$). We then apply the equivariant version of Lillig's theorem (cf. [7] for the non-equivariant version), and deduce that $B_0^n A \hookrightarrow B^n A$ is a G-cofibration for every positive integer n. $\qquad\square$

Corollary C.4. *For every $\Gamma - G$-space and every positive integer n, the projection $B^n A \to B^n A / B_0^n A$ is an equivariant pointed homotopy equivalence.*

For every $k \in [n]$, the composite maps

$$\left(\underline{A}(\mathbf{i_1} \times \cdots \times \mathbf{i_n}) \times (\Delta^{i_1} \times \cdots \times \Delta^{i_n})\right) \times I \longrightarrow \underline{A}(\mathbf{i_1} \times \cdots \times \mathbf{i_k} \times \mathbf{1} \times \cdots \times \mathbf{i_n}) \times (\Delta^{i_1} \times \cdots \times \Delta^{i_k} \times \Delta^1 \times \cdots \times \Delta^{i_n})$$
$$\longrightarrow B^{n+1}\underline{A} \longrightarrow B^{n+1}\underline{A}/j^k_{n+1}(B^n\underline{A}(\mathbf{0})),$$

for $(i_1, \ldots, i_n) \in \mathbb{N}^n$, give rise to a pointed G-map $B^n A \times I \longrightarrow B^{n+1}A/j^k_{n+1}(B^n A(\mathbf{0}))$.

For every $k \in [n]$, this last map yields a continuous map $\Sigma B^n A \longrightarrow B^{n+1}A/\left(\underset{k \in [n]}{\cup} j^k_{n+1}(B^n\underline{A}(\mathbf{0}))\right)$

and, furthermore, a map $B^n A \longrightarrow \Omega(B^{n+1}A/B^n_0 A)$. If s denotes an homotopy inverse of the projection $B^{n+1}A \to B^{n+1}A/B^n_0 A$, composing the preceding map with $\Omega(s)$ yields, for every $k \in [n]$, a pointed G-map

$$\underline{i}^k_n : B^n A \longrightarrow \Omega B^{n+1} A$$

which is uniquely defined up to an equivariant pointed homotopy.

In the spirit of Segal's article [14], the morphism $\underline{i}^0_n$ is defined solely by considering the continuous map $B^n A \to \Omega(B^{n+1}A/j^0_{n+1}(B^n\underline{A}(\mathbf{0})))$ defined earlier in our construction, and by composing it with a homotopy inverse of $\Omega(B^{n+1}A) \to \Omega(B^{n+1}A/j^0_n(B^n\underline{A}(\mathbf{0})))$ (such an inverse exists because j^0_n is a cofibration and $B^n\underline{A}(\mathbf{0})$ is G-contractible). It is now clear that the morphism we have just constructed is the same one as Segal's, as a morphism in the category $CG^{h\bullet}_G$.

Let Σ_n denote the group of permutations of $\{1, \ldots, n\}$, and let $\sigma \in \Sigma_n$. Then σ gives rise to $\underline{\sigma} : \begin{cases} I^n/\partial I^n & \to I^n/\partial I^n \\ (x_1, \ldots, x_n) & \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \end{cases}$, and furthermore, for every pointed G-space X, to an equivariant homeomorphism

$$\overline{\sigma}_X : \begin{cases} \Omega^n X & \longrightarrow \Omega^n X \\ (\varphi : S^n \to X) & \longmapsto (\varphi \circ (\underline{\sigma})^{-1} : S^n \to X). \end{cases}$$

For $0 \le k \le n$, we let $\sigma_k \in \Sigma_{n+1}$ denote the permutation of $\{1, \ldots, n+1\}$ which acts trivially on $\{1, \ldots, n-k\}$, and whose restriction to $\{n-k+1, \ldots, n+1\}$ is the decreasing cycle (i.e. $(n+1, n, n-1, \ldots, n-k+1)$). We finally set

$$i^k_n := \overline{(\sigma_k)}_{B^{n+1}A} \circ \Omega^n(\underline{i}^k_n) : \Omega^n B^n A \longrightarrow \Omega^{n+1} B^{n+1} A.$$

From there, the properties of the i^k_n's are similar to those claimed in [9] and are proven with similar arguments. We shall only restate them here:

Proposition C.1. *For every positive integer n and every $k \in [n]$, the morphism i^k_n is a G-weak equivalence.*

Proposition C.2. *For any fixed positive integer n, the maps $i^0_n, i^1_n, \ldots, i^n_n$ all define the same morphism in the category $CG^{h\bullet}_G[W^{-1}_G]$.*

Remarks 21. (i) For any positive integer n, the maps i^k_n are morphisms between equivariant H-spaces.

(ii) For any Lie group G, and any $m \in \mathbb{N}^* \cup \{\infty\}$, $\underline{\mathrm{Vec}}^{F,m}_G(\mathbf{0}) = *$. This justifies that the previous results can be applied to the $\Gamma - G$-space $\underline{\mathrm{Vec}}^{F,m}_G$, as is done in Section 4.5.

Acknowledgements

I would like to thank Bob Oliver for his constant support and the countless good advice he gave me during my research on this topic.

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